Learning Objectives

By the end of this worksheet, you will:

- Prove statements using the definition of Big-Oh and its negation.
- Represent constant functions in Big-Oh expressions.
- Understand and use the definition of Omega and Theta to compare functions.

For your reference, here is the formal definition of Big-Oh:

\[ g \in \mathcal{O}(f) : \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, \; n \geq n_0 \Rightarrow g(n) \leq cf(n) \]

1. **Constant functions.** As we discussed in class, constant functions, like \( f(n) = 100 \), will play an important role in our analysis of running time next week. For now let’s get comfortable with the notation.

   (a) Let \( g : \mathbb{N} \to \mathbb{R}^{\geq 0} \). Show how to express the statement \( g \in \mathcal{O}(1) \) by expanding the definition of Big-Oh. \(^1\)

   **Solution**
   
   \[ g \in \mathcal{O}(1) : \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, \; n \geq n_0 \Rightarrow g(n) \leq c. \]

   (b) Prove that \( 100 + \frac{77}{n+1} \in \mathcal{O}(1) \).

   Note: this proof isn’t too mathematically complex; treat this as another exercise in making sure you understand the definition of Big-Oh!

   **Hint:** one algebraic property you can use is that \( \forall x, y \in \mathbb{R}^+, \; x \geq y \Rightarrow \frac{1}{x} \leq \frac{1}{y} \).

   **Solution**
   
   We want to prove that \( \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, \; n \geq n_0 \Rightarrow 100 + \frac{77}{n+1} \leq c \).

   There are many possible choices of \( c \) and \( n_0 \) here. One possibility is \( c = 101 \) and \( n = 76 \). We leave the calculation as an exercise.

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\(^1\) Remember that we often abbreviate Big-Oh expressions to just show the function bodies. “\( \mathcal{O}(1) \)” is really shorthand for “\( \mathcal{O}(f) \), where \( f \) is the constant function \( f(n) = 1 \).”
2. **Omega.** Recall that we can think of Big-Oh notation as describing an *upper bound* on the rate of growth of a function: saying “$g \in O(f)$” is like saying “$g$ grows at most as fast as $f$.” As we saw in class, sometimes we care just as much about a *lower bound* on the rate of growth and for this, we have the symbol $\Omega$ (the Greek letter Omega), which is defined analogously to Big-Oh:

$$g \in \Omega(f) : \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \geq cf(n)$$

Using this definition, prove that for all $f, g : \mathbb{N} \to \mathbb{R}^\geq 0$, if $g \in O(f)$, then $f \in \Omega(g)$.

**Solution**

*Proof.* Let $f, g : \mathbb{N} \to \mathbb{R}^\geq 0$. Assume that $g \in O(f)$, i.e., that there exist $c_1, n_1 \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, if $n \geq n_1$ then $g(n) \leq c_1 f(n)$. We want to prove that there exist $c_2, n_2 \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, if $n \geq n_2$ then $f(n) \geq c_2 g(n)$.

Let $c_2 = \frac{1}{c_1}$ and $n_2 = n_1$. Let $n \in \mathbb{N}$, and assume that $n \geq n_2$. We want to prove that $f(n) \geq c_2 g(n)$.

Since $n_2 = n_1$, we know from our assumption that $n \geq n_1$. So then by our first assumption (that $g \in O(f)$), we know that $g(n) \leq c_1 f(n)$. Dividing both sides by $c_1$ yields $\frac{1}{c_1} g(n) \leq f(n)$, and so $c_2 g(n) \leq f(n)$.  \[\square\]
3. **Theta.** As we saw in class, both Big-Oh and Omega are limited in the same way as inequalities on numbers. 

“$2 \leq 10^{10}$” is a true statement, but not very insightful; similarly, “$n + 1 \in \mathcal{O}(n^{10})$” and “$2^n + n^2 \in \Omega(n)$” are both true, but not very precise.

Our final piece of notation is the symbol $\Theta$ (the Greek letter Theta), which we defined in rather simple terms:

$$g \in \Theta(f) : g \in \mathcal{O}(f) \land g \in \Omega(f)$$

Or equivalently,

$$g \in \Theta(f) : \exists c_1, c_2, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow c_1 f(n) \leq g(n) \leq c_2 f(n)$$

As we discussed, when we write $g \in \Theta(f)$, what we mean is “$g$ grows at most as quickly as $f$ and $g$ grows at least as quickly as $f$”—in other words, that $f$ and $g$ have the same rate of growth. In this case, we call $f$ a **tight bound** on $g$, since $g$ is essentially squeezed between constant multiples of $f$.

Prove that for all functions $g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$, and all numbers $a \in \mathbb{R}^{\geq 0}$, if $g \in \Omega(1)$, then $a + g \in \Theta(g)$.

[Or in other words, for such functions $g$, shifting them by a constant amount does not change their “Theta” bound.]

**Solution**

**Proof.** Let $g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$, and let $a \in \mathbb{R}^{\geq 0}$. Assume that $g \in \Omega(1)$, i.e., that there exist $c_0, n_0 \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, if $n \geq n_0$ then $g(n) \geq c_0$. We want to prove that $a + g \in \Theta(g)$, i.e., that there exist $c_1, c_2, n_1 \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, if $n \geq n_1$ then $c_1 g(n) \leq a + g(n) \leq c_2 g(n)$.

Let $c_1 = 1$, $c_2 = \frac{a}{c_0} + 1$, and $n_1 = n_0$. Let $n \in \mathbb{N}$, and assume that $n \geq n_1$. We want to prove that $c_1 g(n) \leq a + g(n) \leq c_2 g(n)$.

[We leave the calculation as an exercise. The trickiest part was figuring out how to choose $c_2$; the intuition is that we need to take the assumed inequality $g(n) \geq c_0$ and turn the right-hand side into $a$ instead of $c_0$.]  

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2 Here we use $a + g$ to denote the function $g_1$ defined as $g_1(n) = a + g(n)$ for all $n \in \mathbb{N}$.  

4. **Negating Big-Oh.** So far, we have only looked at proving that a function is Big-Oh of another function. In this question, we’ll investigate what it means to show that a function isn’t Big-Oh of another.

(a) Express the statement \( g \notin O(f) \) in predicate logic, using the expanded definition of Big-Oh. (As usual, simplify so that all negations are pushed as far “inside” as possible.)

**Solution**

\[
g \notin O(f) : \forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, \ n \geq n_0 \land g(n) > cf(n)
\]

(b) Prove that for all positive real numbers \( a \) and \( b \), if \( a > b \) then \( n^a \notin O(n^b) \).

**Hint:** for all positive real numbers \( x \) and \( y \), \( x > y \iff \log x > \log y \).

**Solution**

**Proof.** Let \( a, b \in \mathbb{R}^+ \), and assume that \( a > b \). We want to show the following:

\[
\forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, \ n \geq n_0 \land n^a > cn^b
\]

Let \( c, n_0 \in \mathbb{R}^+ \), Let \( n = \left\lceil n_0 + c^{1/(a-b)} \right\rceil \). We want to prove that \( n \geq n_0 \) and \( n^a > cn^b \).

[We leave the rest of the proof as an exercise.]

*The ceiling function in the choice of \( n \) is used to ensure that \( n \) is a natural number. We chose this value of \( n \) because we want to ensure that \( n \geq n_0 \), and that \( n \geq c^{1/(a-b)} \).