Learning Objectives

By the end of this worksheet, you will:

- Prove and disprove statements using the definition of Big-Oh.
- Investigate properties of Big-Oh for some common families of functions.

Note: In Big-Oh expressions, it will be convenient to just write down the “body” of the functions rather than defining named functions all the time. We’ll always use the variable \( n \) to represent the function input, and so when we write “\( n \in \mathcal{O}(n^2) \),” we really mean “the functions defined as \( f(n) = n \) and \( g(n) = n^2 \) satisfy \( f \in \mathcal{O}(g) \).”

As a reminder, here is the formal definition of Big-Oh:

\[
g \in \mathcal{O}(f) : \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq cf(n) \quad \text{where } f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}
\]

1. Comparing polynomials. Our first step in comparing different families of functions is looking at different powers of \( n \). Consider the following statement, which generalizes the fact that \( n \in \mathcal{O}(n^2) \):

\[
\forall a, b \in \mathbb{R}^+, a \leq b \Rightarrow n^a \in \mathcal{O}(n^b)
\]

(a) Rewrite the above statement by expanding the definition of Big-Oh.

Solution

\[
\forall a, b \in \mathbb{R}^+, a \leq b \Rightarrow \left( \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow n^a \leq cn^b \right)
\]

(b) Prove the above statement. **Hint:** you can actually pick \( c \) and \( n_0 \) to both be 1. Even though this is pretty simple, take the time to write the formal proof as a good warm-up for the rest of this worksheet.

Solution

**Proof.** Let \( a, b \in \mathbb{R}^+ \), and assume \( a \leq b \). Let \( c = 1 \) and \( n_0 = 1 \). Let \( n \in \mathbb{N} \), and assume that \( n \geq n_0 \). We want to prove that \( n^a \leq n^b \).

We can start with our assumption that \( a \leq b \) and calculate:

\[
a \leq b \\
n^a \leq n^b \quad \text{(since } n \geq 1) \\
n^a \leq cn^b \quad \text{(since } c = 1)
\]

Note: going from \( a \leq b \) to \( n^a \leq n^b \) involves raising \( n \) to the power of both sides. This is valid when \( n \geq 1 \).

2. Comparing logarithms. One slight oddity about the definition of Big-Oh is that it treats all logarithmic functions “the same”. Your task in this question is to investigate this by proving the following statement:

\[
\forall a, b \in \mathbb{R}^+, a > 1 \land b > 1 \Rightarrow \log_a n \in \mathcal{O}(\log_b n)
\]

We won’t ask you to expand the definition of Big-Oh, but if you aren’t quite sure, then we highly recommend doing so before attempting even your rough work.

**Hint:** use the “change of base rule” for logarithms.

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1 If you are concerned by the fact that \( \log n \) is not defined at \( n = 0 \), you can replace \( \log_a n \) with \( \log_a (1 + n) \) in the above, and similarly with \( \log_b \). We usually don’t worry about this subtlety, since our concern is with the value of the functions for larger values of \( n \). Picking an \( n_0 > 0 \) avoids the evaluation worry.
Solution

Proof. Let \( a, b \in \mathbb{R}^+ \). Assume that \( a > 1 \) and \( b > 1 \). Let \( n_0 = 1 \), and let \( c = \frac{1}{\log_b a} \). Let \( n \in \mathbb{N} \), and assume that \( n \geq n_0 \). We want to show that \( \log_a n \leq c \cdot \log_b n \).

The \textit{logarithm change of base rule} tells us the following: \[ \forall a, b, x \in \mathbb{R}^+, a \neq 1 \land b \neq 1 \Rightarrow \log_a x = \frac{\log_b x}{\log_b a} \]

Using this rule, we can write:

\[
\log_a n = \frac{\log_b n}{\log_b a} = \frac{1}{\log_b a} \log_b n = c \cdot \log_b n
\]

Since we’ve proved that \( \log_a n = c \cdot \log_b n \), we can conclude that \( \log_a n \leq c \cdot \log_b n \).

[Note: we didn’t need to use the assumption that \( n \geq 1 \) in this proof.]

\[ \star \] Since \( a, b > 1 \), we know that \( c > 0 \).

\[ \dagger \] When the bases are equal to 1, \( \log_a x \) is undefined when \( x \neq 1 \).
Now let’s look at one of the most important properties of Big-Oh: how it behaves when adding functions together. Let \( f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0} \). We define the sum of \( f \) and \( g \) as the function \( f + g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0} \) such that \( \forall n \in \mathbb{N}, (f + g)(n) = f(n) + g(n) \). For example, if \( f(n) = 2n \) and \( g(n) = n^2 + 3 \), then \( (f + g)(n) = 2n + n^2 + 3 \).

Consider the following statement:

\[
\forall f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}, \ g \in \mathcal{O}(f) \Rightarrow f + g \in \mathcal{O}(f)
\]

In other words, if \( g \) is Big-Oh of \( f \), then \( f + g \) is no bigger than just \( f \) itself, asymptotically speaking.

Your task for this question is to prove this statement. Keep in mind this is an implication: you’re going to assume that \( g \in \mathcal{O}(f) \), and you want to prove that \( f + g \in \mathcal{O}(f) \). It will likely be helpful to write out the full statement (with the definition of Big-Oh expanded), and use subscripts to help keep track of the variables.

**Solution**

Here’s the full statement, with the definitions expanded:

\[
\forall f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}, \ (\exists c, n_0 \in \mathbb{R}^{+}, \ \forall n \in \mathbb{N}, \ n \geq n_0 \Rightarrow g(n) \leq cf(n)) \Rightarrow \\
(\exists c_1, n_1 \in \mathbb{R}^{+}, \ \forall n \in \mathbb{N}, \ n \geq n_1 \Rightarrow f(n) + g(n) \leq c_1f(n))
\]

**Proof.** Let \( f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0} \). Assume that \( g \in \mathcal{O}(f) \), i.e., there exist \( n_0, c \in \mathbb{R}^{+} \) such that for all natural numbers \( n \), if \( n \geq n_0 \) then \( g(n) \leq cf(n) \). We want to prove that \( f + g \in \mathcal{O}(f) \).

Let \( n_1 = n_0 \), and \( c_1 = c + 1 \). Let \( n \in \mathbb{N} \), and assume that \( n \geq n_1 \). We want to prove that \( f(n) + g(n) \leq c_1f(n) \).

Since \( n \geq n_1 = n_0 \), by our assumption we know that \( g(n) \leq cf(n) \). So then:

\[
\begin{align*}
g(n) &\leq cf(n) \\
f(n) + g(n) &\leq f(n) + cf(n) \\
f(n) + g(n) &\leq (c + 1)f(n) \\
f(n) + g(n) &\leq c_1f(n)
\end{align*}
\]