Learning Objectives

By the end of this worksheet, you will:

- Write proofs using simple induction with different starting base cases.
- Write proofs using simple induction within the scope of a larger proof.

1. **Induction (summations).** If marbles are arranged to form an equilateral triangle shape, with \( n \) marbles on each side, a total of \( \sum_{i=1}^{n} i \) marbles will be required. (For convenience, we also define \( T_0 = 0 \).)

In lecture, we proved that \( \sum_{i=1}^{n} i = n(n+1)/2 \). For each \( n \in \mathbb{N} \), let \( T_n = n(n+1)/2 \); these numbers are usually called the **triangular numbers**. Use induction to prove that

\[
\forall n \in \mathbb{N}, \sum_{j=0}^{n} T_j = \frac{n(n+1)(n+2)}{6}
\]
2. **Induction (inequalities).** Consider the statement:

For every positive real number $x$ and every natural number $n$, $(1 + x)^n \geq (1 + nx)$.

We can express the statement using the notation of predicate logic as:

$$\forall x \in \mathbb{R}^+, \forall n \in \mathbb{N}, \ (1 + x)^n \geq 1 + nx$$

Notice that in this statement, there are two universally-quantified variables: $n$ and $x$. Prove the statement is true using the following approach:

(a) Use the standard proof structure to introduce $x$.
(b) When proving the $(\forall n \in \mathbb{N}, (1 + x)^n \geq 1 + nx)$, do induction on $n$.

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1. For extra practice, think about the following questions. First, would the statement still be correct with the order of the quantifiers reversed: $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}^+, \ (1 + x)^n \geq 1 + nx$? Second, if this variation is correct, how would this change the proof?

2. Your predicate $P(n)$ that you want to prove will also contain the variable $x$—that’s okay, since when we do the induction proof, $x$ has already been defined.
3. **Changing the starting number.** Recall that you previously proved that \( \forall n \in \mathbb{N}, n \leq 2^n \) using induction.

   (a) First, use trial and error to fill in the blank to make the following statement true – try finding the *smallest natural number* that works!
   \[
   \forall n \in \mathbb{N}, n \geq \, \text{[blank]} \Rightarrow 30n \leq 2^n
   \]

   (b) Now, prove the completed statement using induction. Be careful about how you choose your base case!
(c) Extra. Using what you’ve learned in this worksheet, prove the following generalization of the previous statement:

\[ \forall a \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, \ n \geq n_0 \Rightarrow an \leq 2^n \]

(Note: it’s easier if you don’t try to find the smallest possible \( n_0 \).)