Learning Objectives

By the end of this worksheet, you will:

- Analyse the average-case running time of an algorithm.

1. **Average-case analysis.** Consider the following algorithm that we studied a few weeks ago. The input is an array $A$ of length $n$, containing a list of $n$ integers.

   ```python
def has_even(A: List[int]) -> int:
    n = len(A)
    for i in range(n):
        if A[i] % 2 == 0:
            return i
    return -1
```

We proved that the worst-case running time of this algorithm is $\Theta(n)$. In this problem we will analyse its average-case running time.

For this analysis, we will consider the set of binary lists $A$ of length $n$. That is, $A$ is a list of $n$ integers, where each integer is either 0 or 1.

(a) For each $n \in \mathbb{Z}^+$, let $\mathcal{I}_n$ be the set of all binary lists of length $n$. Find an expression (in terms of $n$) for $|\mathcal{I}_n|$, the size of $\mathcal{I}_n$. 
(b) For each \( n \in \mathbb{Z}^+ \) and each \( i \in \{0, 1, \ldots, n - 1\} \), let \( S_n(i) \) denote the set of all binary lists \( A \) such that the first 0 occurs in position \( i \). More precisely, every list in \( S_n(i) \) satisfies the following two properties:

(i) \( A[i] = 0 \).

(ii) for all \( j \in \mathbb{N} \), if \( j < i \) then \( A[j] = 1 \).

Also let \( S_n(n) \) be the set of binary lists that contain no 0's. For each \( i, 0 \leq i \leq n \), find an expression for \( |S_n(i)| \).

(c) Argue that for every \( n \in \mathbb{Z}^+ \), each binary list of length \( n \) is in exactly one set \( S_n(i) \) (for some \( i \in \{0, \ldots, n\} \)).

Use this to show that \( \sum_{i=0}^{n} |S_n(i)| = |I_n| \).
(d) Let the runtime of the algorithm on a binary list $A$ be the number of iterations of the loop. Give an exact expression for the average runtime of the above algorithm using the quantities that you calculated. You should get a summation; do not simplify the summation in this part.

(e) Show that the runtime that you calculated is in $\mathcal{O}(1)$. You may use without proof that for all $x \in \mathbb{R}$, if $|x| < 1$, then $\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}$. 