Learning Objectives

By the end of this worksheet, you will:

- Analyse the average-case running time of an algorithm.

1. **Average-case analysis.** Consider the following algorithm that we studied a few weeks ago. The input is an array $A$ of length $n$, containing a list of $n$ integers.

```python
def has_even(lst: List[int]) -> int:
    n = len(lst)
    for i in range(n):
        if lst[i] % 2 == 0:
            return i
    return -1
```

We proved that the worst-case running time of this algorithm is $\Theta(n)$. In this problem we will analyse its average-case running time.

For this analysis, we will consider the sets of *binary* lists $lst$ of length $n$, for each $n \in \mathbb{Z}^+$. That is, $lst$ is a list of $n$ integers, where each integer is either 0 or 1.

(a) For each $n \in \mathbb{Z}^+$, let $I_n$ be the set of all binary lists of length $n$. Find an expression (in terms of $n$) for $|I_n|$, the size of $I_n$. 


(b) For each \( n \in \mathbb{Z}^+ \) and each \( i \in \{0, 1, \ldots, n - 1\} \), let \( S_{n,i} \) denote the set of all binary lists \( lst \) of length \( n \) where the first 0 occurs in position \( i \). More precisely, every list \( lst \) in \( S_{n,i} \) satisfies the following two properties:

(i) \( lst[i] = 0 \).

(ii) for all \( j \in \mathbb{N} \), if \( j < i \) then \( lst[j] = 1 \).

For each \( i \), \( 0 \leq i \leq n \), find an expression for \( |S_{n,i}| \).

(c) Also, for each \( n \in \mathbb{Z}^+ \), let \( S_{n,n} \) denote the set of binary lists of length \( n \) that do not contain a 0 at all. Find an expression for \( |S_{n,n}| \).

(d) Give a brief argument (informal proof) that for every \( n \in \mathbb{Z}^+ \), each binary list of length \( n \) is in exactly one set \( S_{n,i} \) (for some \( i \in \{0, 1, \ldots, n\} \)). That is, you're arguing that \( S_{n,0}, S_{n,1}, \ldots, S_{n,n} \) form a partition of \( \mathcal{I}_n \). 
(e) Assume that we calculate the running time of `has_even` by counting just the costs of Lines 4 and 7. Find an exact expression for the average runtime of this algorithm for this input set $I_n$, in terms of $n$.

You should get a summation; do not simplify the summation in this part.

(f) Show that the average running time expression that you calculated is in $O(1)$. You may use the fact that for all $x \in \mathbb{R}$, if $|x| < 1$, then $\sum_{i=1}^{\infty} ix^i = \frac{x}{(1 - x)^2}$. 