Learning objectives

By the end of this worksheet, you will:

- Understand and apply definitions about sets, strings, and common mathematical functions.
- Simplify summation and product expressions.

1. Set complement. Let $A$ and $U$ be sets, and assume that $A \subseteq U$. The complement of $A$ in $U$, denoted $A^c$, is defined to be set of elements that are in $U$ but not $A$. $A^c$ depends on the choice of both $U$ and $A$!

(a) Let $U$ be the set of natural numbers between 1 and 6, inclusive. Let $A = \{2, 5\}$. What is $A^c$?

**Solution**

\[ A^c = \{1, 3, 4, 6\} \]

(b) Given an arbitrary $A$ and $U$, write an expression for $A^c$ in terms of $A$, $U$, and the set difference operator \(\setminus\).

**Solution**

\[ A^c = U \setminus A \]

(c) Let $U = \mathbb{R}$, $A = \{x \mid x \in U \text{ and } 0 < x \leq 2\}$, and $B = \{x \mid x \in U \text{ and } 1 \leq x < 4\}$. Find each of the following: $A^c \cap B^c$, $A^c \cup B^c$, $(A \cap B)^c$ and $(A \cup B)^c$. Drawing number lines may be helpful. What relationships do you notice between these sets?

**Solution**

\[ A^c \cap B^c = \{x \mid x \in U \text{ and } x \leq 0 \text{ or } x \geq 4\} \]
\[ A^c \cup B^c = \{x \mid x \in U \text{ and } x < 1 \text{ or } x > 2\} \]
\[ (A \cap B)^c = \{x \mid x \in U \text{ and } x < 1 \text{ or } x > 2\} \]
\[ (A \cup B)^c = \{x \mid x \in U \text{ and } x \leq 0 \text{ or } x \geq 4\} \]

There are two equalities we observe: $A^c \cap B^c = (A \cup B)^c$ and $A^c \cup B^c = (A \cap B)^c$ (de Morgan’s laws for sets).

2. Set partitions. Let $A$ be a set. A (finite or infinite) collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is called a partition of $A$ when (1) $A$ is the union of all of the $A_i$ and (2) the sets $A_1, A_2, A_3, \ldots$ do not have any element in common.

(a) Recall that $\mathbb{Z}^+$ is the set of all positive integers. Let

\[ T_0 = \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k, \text{ for some integer } k\}, \]
\[ T_1 = \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k + 1, \text{ for some integer } k\}, \]
\[ T_2 = \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k + 2, \text{ for some integer } k\}, \]
\[ T_3 = \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 6k, \text{ for some integer } k\}. \]

Write the smallest three elements of $T_0$, of $T_1$, of $T_2$, and of $T_3$.

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1 We say the $A_i$ are exhaustive.
2 We say the $A_i$ are mutually disjoint (or pairwise disjoint or non-overlapping) when no two distinct sets $A_i$ and $A_j$ have any element in common.
Solution
\[ T_0 = \{3, 6, 9, \ldots\}, \ T_1 = \{1, 4, 7, \ldots\}, \ T_2 = \{2, 5, 8, \ldots\}, \ T_3 = \{6, 12, 18, \ldots\}. \]

(b) Write down a partition of \( \mathbb{Z}^+ \) using \( T_0, T_1, T_2, \) and/or \( T_3. \) Why can’t you use all four sets?

Solution

The set \( \{T_0, T_1, T_2\} \) is a partition of \( \mathbb{Z}^+ \), since, when any positive integer is divided by 3, the possible integer remainders are 0, 1, and 2. The sets \( T_0, T_1, T_2 \) list the numbers whose remainder when divided by 3 are 0, 1, or 2, respectively.

\( T_3 \subseteq T_0, \) so we can’t use both \( T_0 \) and \( T_3 \) in our partition (they have elements in common).
3. **Strings.** An alphabet $A$ is a set of symbols like $\{0,1\}$ or $\{a,b,c\}$. We define a string over alphabet $A$ as an ordered sequence of elements from $A$; the length of a finite string is its number of elements.

For example, $011$ is a string over $\{0,1\}$ of length three, and $abbbacc$ is a string over $\{a,b,c\}$ of length seven.

(a) Write down all strings over the alphabet $\{0,1\}$ of length three (you should have eight in total).

**Solution**

\{000, 001, 010, 011, 100, 101, 110, 111\}

(b) Let $S_1$ be the set of all strings over $\{a,b,c\}$ that have length two, and $S_2$ be the set of all strings over $\{a,b,c\}$ that start and end with the same letter. Find $S_1 \cap S_2$ and $S_1 \setminus S_2$.

**Solution**

$S_1 \cap S_2 = \{aa, bb, cc\}$.

$S_1 \setminus S_2 = \{ab, ac, ba, bc, ca, cb\}$.

Note that $S_2$ is actually an infinite set, but both $S_1 \cap S_2$ and $S_1 \setminus S_2$ are finite.

(c) What is the relationship between $S_1$, $S_1 \cap S_2$, and $S_1 \setminus S_2$?

**Solution**

Hint: look at $(S_1 \cap S_2) \cup (S_1 \setminus S_2)$.

4. **The floor and ceiling functions.** Let $x \in \mathbb{R}$. We define the **floor of** $x$, denoted $\lfloor x \rfloor$, to be the largest integer that is less than or equal to $x$. Similarly, we define the **ceiling of** $x$, denoted $\lceil x \rceil$, to be the smallest integer that is greater than or equal to $x$.

(a) Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the following values of $x$: $x = \frac{25}{4}$, $x = 0.999$, and $x = -2.01$.

**Solution**

$\lfloor \frac{25}{4} \rfloor = \lfloor 6.25 \rfloor = 6$, $\lceil \frac{25}{4} \rceil = \lceil 6.25 \rceil = 7$, $\lfloor 0.999 \rfloor = 0$, $\lceil 0.999 \rceil = 1$, $\lfloor -2.01 \rfloor = -3$, $\lceil -2.01 \rceil = -2$.

(b) What is the domain and codomain of the floor and ceiling functions?

**Solution**

The domain is $\mathbb{R}$ and the codomain is $\mathbb{Z}$.

(c) Consider the following statement: For all real numbers $x$ and $y$, $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$. Is this statement is True or False? Why?

**Solution**

The statement is False, since, for example, $\lfloor \frac{1}{2} + \frac{2}{3} \rfloor = \lfloor \frac{7}{6} \rfloor = 1$, while $\lfloor \frac{1}{2} \rfloor + \lfloor \frac{2}{3} \rfloor = 0 + 0 = 0$. 
5. **Sum and product notation.** Recall that the notation $\sum_{i=j}^{k} f(i)$ gives us a short form for $f(j) + f(j+1) + \cdots + f(k-1) + f(k)$, and that $\prod_{i=j}^{k} f(i)$ gives us a short form for $f(j) \times f(j+1) \times \cdots \times f(k-1) \times f(k)$.

(a) Expand the following expressions into their long sum/product form. Do not evaluate the resulting expressions.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Long Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{k=1}^{3} (k + 1)$</td>
<td>$(1 + 1) + (2 + 1) + (3 + 1)$</td>
</tr>
<tr>
<td>$\sum_{k=-1}^{2} (k^2 + 3)$</td>
<td>$(1 + 3) + (0 + 3) + (1 + 3) + (4 + 3)$</td>
</tr>
<tr>
<td>$\sum_{k=1}^{5} (2k)$</td>
<td>$2 + 4 + 6 + 8 + 10$</td>
</tr>
<tr>
<td>$\sum_{m=0}^{1} \frac{1}{2^m}$</td>
<td>$\frac{1}{2^0} + \frac{1}{2^1}$</td>
</tr>
<tr>
<td>$\sum_{j=0}^{4} \frac{j}{j+1}$</td>
<td>$0 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5}$</td>
</tr>
<tr>
<td>$\prod_{i=2}^{4} \frac{i(i+2)}{(i-1)(i+1)}$</td>
<td>$2 \cdot \frac{4}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{6}{5}$</td>
</tr>
</tbody>
</table>

(b) Rewrite each of the following expressions by using sum or product notation.

**Solution**

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>$3 + 6 + 12 + 24 + 48 + 96 = \sum_{i=0}^{5} 3 \cdot 2^i$</td>
<td>$\frac{1}{3} + \frac{4}{9} + \frac{9}{27} + \frac{16}{81} + \frac{25}{243} + \frac{36}{729} = \sum_{j=1}^{6} \frac{j^2}{3^j}$</td>
</tr>
<tr>
<td>$0 + 1 - 2 + 3 - 4 + 5 = \sum_{j=0}^{5} (-1)^{j+1} \cdot j$</td>
<td>$\left(\frac{1}{1+1}\right) \times \left(\frac{2}{2+1}\right) \times \left(\frac{3}{3+1}\right) \times \cdots \times \left(\frac{k}{k+1}\right) = \prod_{j=1}^{k} \left(\frac{j}{j+1}\right)$</td>
</tr>
<tr>
<td>$\left(\frac{1 \cdot 2}{3 \cdot 4}\right) \times \left(\frac{2 \cdot 3}{4 \cdot 5}\right) \times \left(\frac{3 \cdot 4}{5 \cdot 6}\right)$</td>
<td>$\prod_{j=1}^{3} \frac{j \cdot (j+1)}{(j+2) \cdot (j+3)}$</td>
</tr>
</tbody>
</table>
6. **Sum and product laws.** It is possible to prove properties that help us manipulate sums and products. Let $m, n \in \mathbb{Z}$, and let $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ be sequences of real numbers, and let $c \in \mathbb{R}$. Then the following equations hold:

$$
\sum_{i=m}^{n} (a_i + b_i) = \left( \sum_{i=m}^{n} a_i \right) + \left( \sum_{i=m}^{n} b_i \right) \quad \text{(separating sums)}
$$

$$
\prod_{i=m}^{n} (a_i \cdot b_i) = \left( \prod_{i=m}^{n} a_i \right) \cdot \left( \prod_{i=m}^{n} b_i \right) \quad \text{(separating products)}
$$

$$
\sum_{i=m}^{n} c \cdot a_i = c \cdot \left( \sum_{i=m}^{n} a_i \right) \quad \text{(pulling out constant)}
$$

$$
\sum_{i=m}^{n} a_i = \sum_{i'=0}^{n-m} a_{i'+m} \quad \text{(changing index)}
$$

$$
\prod_{i=m}^{n} a_i = \prod_{i'=0}^{n-m} a_{i'+m} \quad \text{(changing index)}
$$

Using these laws, rewrite each of the following as a single sum or product, but do not evaluate your final sum/product.

$$
3 \cdot \sum_{i=1}^{n} (2i - 3) + \sum_{i=1}^{n} (4 - 5i)
$$

**Solution**

$$
3 \cdot \sum_{i=1}^{n} (2i - 3) + \sum_{i=1}^{n} (4 - 5i) = \sum_{i=1}^{n} (6i - 9) + \sum_{i=1}^{n} (4 - 5i)
$$

$$
= \sum_{i=1}^{n} ((6i - 9) + (4 - 5i))
$$

$$
= \sum_{i=1}^{n} (i - 5)
$$

$$
\left( \prod_{i=1}^{n} \frac{i}{i+1} \right) \left( \prod_{i=1}^{n} \frac{i+1}{i+2} \right)
$$

**Solution**

$$
\left( \prod_{i=1}^{n} \frac{i}{i+1} \right) \left( \prod_{i=1}^{n} \frac{i+1}{i+2} \right) = \prod_{i=1}^{n} \frac{i}{i+1} \cdot \frac{i+1}{i+2}
$$

$$
= \prod_{i=1}^{n} \frac{i}{i+2}
$$

$$
\sum_{i=10}^{106} 2i + \sum_{i=101}^{15} (i - 1) \text{ (change the indexes to match)}
$$

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3. Because of how we’ve defined the **empty sum** and **empty product**, these equations hold even when $n < m$!

4. We’ll cover some formulas for evaluating common sums and products throughout this course.
Solution

\[
\sum_{i=10}^{15} 2i + \sum_{i=101}^{106} (i - 1) = \sum_{i=0}^{5} 2(i + 10) + \sum_{i=0}^{5} (i + 101 - 1)
\]

\[
= \sum_{i=0}^{5} 2(i + 10) + \sum_{i=0}^{5} (i + 101 - 1)
\]

\[
= \sum_{i=0}^{5} \left( (2(i + 10) + (i + 101 - 1)) \right)
\]

\[
= \sum_{i=0}^{5} (3i + 120)
\]

Note: it is also possible to change the first summation to go from \(i = 101\) to \(106\), or the second summation to go from \(i = 10\) to \(15\).