Learning objectives

By the end of this worksheet, you will:

- Know and apply various definitions for sets, strings, and common mathematical functions.
- Manipulate summation and product expressions.

1. **Set complement.** Let $A$ and $U$ be sets, and suppose that $A \subseteq U$. The complement of $A$ in $U$, denoted $A^c$, is defined to be set of elements that are in $U$ but not $A$. Notice that this depends on the choice of both $U$ and $A$!

   (a) Let $U$ be the set of natural numbers between 1 and 6, inclusive. Let $A = \{2, 5\}$. What is $A^c$?

   (b) Write an expression for $A^c$ that uses the symbols $A$, $U$, and the set difference operator \.

   (c) Let $U = \mathbb{R}$, and consider the sets $A = \{x \mid x \in U \text{ and } 0 < x \leq 2\}$ and $B = \{x \mid x \in U \text{ and } 1 \leq x < 4\}$. Find each of the following: $A^c \cap B^c$, $A^c \cup B^c$, $(A \cap B)^c$ and $(A \cup B)^c$. Drawing number lines may be helpful. Any observations?

2. **Set partitions.** Let $A$ be a set. A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is called a partition of a $A$ when (1) $A$ is the union of all of the $A_i$ and (2) the sets $A_1, A_2, A_3, \ldots$ do not have any element in common.

   (a) Recall that $\mathbb{Z}^+$ is the set of all positive integers. Let

   $$
   T_0 = \{ n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k, \text{ for some integer } k \},
   $$

   $$
   T_1 = \{ n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k + 1, \text{ for some integer } k \},
   $$

   $$
   T_2 = \{ n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k + 2, \text{ for some integer } k \},
   $$

   $$
   T_3 = \{ n \mid n \in \mathbb{Z}^+ \text{ and } n = 6k, \text{ for some integer } k \}.
   $$

   Write the first three elements of $T_0$, of $T_1$, of $T_2$, and of $T_3$.

   (b) Write down a partition of $\mathbb{Z}^+$ using $T_0, T_1, T_2$, and/or $T_3$. Why can’t you use all four sets?

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1 We say the $A_i$ are exhaustive.

2 We say the $A_i$ are mutually disjoint (or pairwise disjoint or non-overlapping) if and only if no two sets $A_i$ and $A_j$ with distinct subscripts have any element in common.
3. **Strings.** An alphabet $A$ is a set of symbols like $\{0, 1\}$ or $\{a, b, c\}$. A string over alphabet $A$ is an ordered sequence of elements from $A$; the length of a finite string is its number of elements.

For example, 011 is a string over $\{0, 1\}$ of length three, and $abbc$ is a string over $\{a, b, c\}$ of length seven.

(a) Write down all strings over the alphabet $\{0, 1\}$ of length three (you should have eight in total).

(b) Let $S_1$ be the set of all strings over $\{a, b, c\}$ that have length two, and $S_2$ be the set of all strings over $\{a, b, c\}$ that start and end with the same letter. Find $S_1 \cap S_2$ and $S_1 \setminus S_2$.

(c) What do you notice about the relationship between $S_1$, $S_1 \cap S_2$, and $S_1 \setminus S_2$?

4. **The floor and ceiling functions.** Given any real number $x$, the **floor of** $x$, denoted $\lfloor x \rfloor$, is defined to be the largest integer that is less than or equal to $x$. Similarly, the **ceiling of** $x$, denoted $\lceil x \rceil$, is defined to be the smallest integer that is greater than or equal to $x$.

(a) What is the domain and range of the floor and ceiling functions?

(b) Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the following values of $x$: $x = \frac{25}{4}$, $x = 0.999$, and $x = -2.01$.

(c) Consider the following statement: For all real numbers $x$ and $y$, $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$. Do you think this statement is True or False? Why?
5. **Sum and Product notation.** Recall that the notation $\sum_{i=j}^{k} f(i)$ gives us a short form for expressing the sum $f(j) + f(j + 1) + \cdots + f(k - 1) + f(k)$, and that $\prod_{i=j}^{k} f(i)$ gives us a short form for expressing the product $f(j) \times f(j + 1) \times \cdots \times f(k - 1) \times f(k)$.

(a) Expand the following expressions into their long sum/product form. Do not evaluate the resulting expressions.

\[
\begin{align*}
\sum_{k=1}^{3} (k+1) & \quad \sum_{n=0}^{1} \frac{1}{2^n} \\
\sum_{k=-1}^{2} (k^2 + 3) & \quad \sum_{j=0}^{4} (-1)^j \frac{j}{j+1} \\
\sum_{k=1}^{5} (2k) & \quad \prod_{i=2}^{4} \frac{i(i+2)}{(i-1)(i+1)}
\end{align*}
\]

(b) Rewrite each of the following expressions by using sum or product notation.

\[
\begin{align*}
3 + 6 + 12 + 24 + 48 + 96 & \quad \frac{1}{3} + \frac{4}{9} + \frac{9}{27} + \frac{16}{81} + \frac{25}{243} + \frac{36}{729} \\
0 + 1 - 2 + 3 - 4 + 5 & \quad \left(\frac{1}{1+1}\right) \times \left(\frac{2}{2+1}\right) \times \left(\frac{3}{3+1}\right) \times \cdots \times \left(\frac{k}{k+1}\right) \\
\left(\frac{1 \cdot 2}{3 \cdot 4}\right) \times \left(\frac{2 \cdot 3}{4 \cdot 5}\right) \times \left(\frac{3 \cdot 4}{5 \cdot 6}\right) & \quad \left(\frac{1}{1+1}\right) \times \left(\frac{2}{2+1}\right) \times \left(\frac{3}{3+1}\right) \times \cdots \times \left(\frac{k}{k+1}\right)
\end{align*}
\]
6. **Sum and product laws.** It is possible to prove results like the following that can be used to help us manipulate sums and products. Let \( m, n \in \mathbb{Z} \), and let \( a_m, a_{m+1}, a_{m+2}, \ldots \) and \( b_m, b_{m+1}, b_{m+2}, \ldots \) be sequences of real numbers, and let \( c \in \mathbb{R} \). Then the following equations hold:

\[
\begin{align*}
\sum_{k=m}^{n} (a_k + b_k) &= \left( \sum_{k=m}^{n} a_k \right) + \left( \sum_{k=m}^{n} b_k \right) \\
\prod_{k=m}^{n} (a_k \cdot b_k) &= \left( \prod_{k=m}^{n} a_k \right) \cdot \left( \prod_{k=m}^{n} b_k \right) \\
\sum_{k=m}^{n} c \cdot a_k &= c \cdot \left( \sum_{k=m}^{n} a_k \right)
\end{align*}
\]

Using these laws, rewrite each of the following as a single sum or product, but do not evaluate your final sum/product. (You’ll learn late in the course how to do so.)

\[
3 \cdot \sum_{k=1}^{n} (2k - 3) + \sum_{k=1}^{n} (4 - 5k) \quad \left( \prod_{k=1}^{n} \frac{k}{k + 1} \right) \left( \prod_{k=1}^{n} \frac{k + 1}{k + 2} \right)
\]

\(^3\)Note that because of how we’ve defined the *empty sum* and *empty product*, these equations hold even when \( n < m \)!