Prep 4 quiz

This is a preview of the published version of the quiz

Started: Jan 23 at 10:01am

Quiz Instructions

Readings

Please read the following part of the Course Notes.

- Chapter 2, pp. 38–55 (up to but not including Characterizations)

General instructions

You can review the general instructions for all prep quizzes at this page. Remember that you can submit multiple times! We have posted a PDF version of the quiz on the course website. You might consider printing this quiz out so that you can work on paper first.

| Question 1 | 1 pts |
Here are several different statements in predicate logic:

1. \( \forall n, k \in \mathbb{N}, \ Q(n, k) \)
2. \( \forall n \in \mathbb{N}, \exists k \in \mathbb{N}, \ Q(n, k) \)
3. \( \forall n \in \mathbb{N}, \ P(n) \)
4. \( \forall n, p \in \mathbb{N}, \ Prime(p) \land p \mid n \Rightarrow Q(n, p) \)
5. \( \forall a, b \in \mathbb{N}, \ \forall n \in \mathbb{N}, \ a \mid n \land b \mid n \Rightarrow P(n) \)
6. \( \forall n \in \mathbb{N}, \ 5 \mid n \land P(n) \)
7. \( \forall n \in \mathbb{N}, \ 5 \mid n \Rightarrow P(n) \)
8. \( \exists a, b \in \mathbb{N}, \ \forall n \in \mathbb{N}, \ a \mid n \land b \mid n \Rightarrow P(n) \)

For each proof header below, select the statement from the list above that is being proved:

Let \( n \) be an arbitrary natural number. Assume that 5 divides \( n \).

Let \( n \) and \( p \) be arbitrary natural numbers. Assume \( p \) is prime and that \( n \) is divisible by \( p \).

Let \( n \) be an arbitrary natural number, and let \( k = 3n + 1 \).

Let \( a = 7 \) and \( b = 9 \). Let \( n \) be an arbitrary natural number, and assume that \( a \) divides \( n \) and that \( b \) divides \( n \).

**Question 2**

Recall the definition of **prime** from lecture, which can be expressed by the following predicate over \( \mathbb{N} \):

\[
Prime(p) : \ p > 1 \land (\forall d \in \mathbb{N}, \ d \mid p \Rightarrow d = 1 \lor d = p)
\]

Suppose we want to prove the following statement: "for every positive integer \( n \), the value \( n^2 + n + 1 \) is prime." What would be the appropriate proof structure for this proof? (Hint: translate the statement into predicate logic first.)
Let \( n \in \mathbb{Z}^+ \). We want to prove that \( n^2 + n + 1 \) is prime, i.e., that \( n^2 + n + 1 > 1 \) and 
\[ \forall d \in \mathbb{N}, \ d \mid (n^2 + n + 1) \Rightarrow d = 1 \lor d = n^2 + n + 1. \]

**Part 1**: we prove that \( n^2 + n + 1 > 1 \).

[Proof body for Part 1...]

**Part 2**: we prove that \( \forall d \in \mathbb{N}, \ d \mid (n^2 + n + 1) \Rightarrow d = 1 \lor d = n^2 + n + 1. \) Let \( d \in \mathbb{N} \), and assume that \( d \mid (n^2 + n + 1) \). We will prove that \( d = 1 \) or that \( d = n^2 + n + 1 \).

[Proof body for Part 2...]

Let \( n \in \mathbb{Z}^+ \). We want to prove that \( n^2 + n + 1 \) is prime, i.e., that \( n^2 + n + 1 > 1 \) and 
\[ \forall d \in \mathbb{N}, \ d \mid (n^2 + n + 1) \Rightarrow d = 1 \lor d = n^2 + n + 1. \]

**Part 1**: we prove that \( n^2 + n + 1 > 1 \).

[Proof body for Part 1...]

**Part 2**: we prove that \( \forall d \in \mathbb{N}, \ d \mid (n^2 + n + 1) \Rightarrow d = 1 \lor d = n^2 + n + 1. \) Let \( d \in \mathbb{N} \). We divide this proof into cases.

**Case 1**: assume that \( d = 1 \).

[Proof body in Case 1...]

**Case 2**: assume that \( d = n^2 + n + 1 \).

[Proof body in Case 2...]

Let \( n \in \mathbb{Z}^+ \). We want to prove that \( n^2 + n + 1 \) is prime, i.e., that 
\[ \forall d \in \mathbb{N}, \ d \mid (n^2 + n + 1) \Rightarrow d = 1 \lor d = n^2 + n + 1. \]

Let \( d \in \mathbb{N} \) and assume that \( d \mid (n^2 + n + 1) \).

We will prove that \( d = 1 \) or that \( d = n^2 + n + 1 \).

[Proof body...]

Let \( n \in \mathbb{Z}^+ \). Assume \( n > 1 \) and \( d \mid n^2 + n + 1 \).

We will prove that \( d = 1 \) or that \( d = n^2 + n + 1 \).

[Proof body...]

**Question 3**

Using the definition of prime from the previous question, select the correct negation of the **Prime** predicate below.

**Hint**: before even looking at the responses, use the negation rules on the definition of the
Suppose we want to prove the following statement: "for every positive integer $n$, the value $n^2 + n + 1$ is not prime."

What would be the appropriate proof structure for this proof? (Hint: translate the statement into predicate logic first!)

- Let $n \in \mathbb{Z}^+$. We will prove that $n^2 + n + 1$ is not prime. Since $n^2 + n + 1 > 1$, we'll need to prove the second part of the "or", i.e., that $\exists d \in \mathbb{N}, \ d \mid (n^2 + n + 1) \land d \neq 1 \land d \neq n^2 + n + 1$.
  
  Let $d = \text{____.}$
  
  **Part 1:** we prove that $d \mid (n^2 + n + 1)$.
  
  [Proof body for Part 1...]
  
  **Part 2:** we prove that $d \neq 1$.
  
  [Proof body for Part 2...]
  
  **Part 3:** we prove that $d \neq n^2 + n + 1$.
  
  [Proof body for Part 3...]
Let $n = 10$. We will prove that $n^2 + n + 1$ is not prime. Since $n^2 + n + 1 > 1$, we'll need to prove the second part of the "or", i.e., that

$$\exists d \in \mathbb{N}, \ d \mid (n^2 + n + 1) \land d \neq 1 \land d \neq n^2 + n + 1.$$  

Let $d = 3$.

**Part 1**: we prove that $d \mid (n^2 + n + 1)$.

**Proof body for Part 1...**

**Part 2**: we prove that $d \neq 1$.

**Proof body for Part 2...**

**Part 3**: we prove that $d \neq n^2 + n + 1$.

**Proof body for Part 3...**

Let $n \in \mathbb{Z}^+$. We will prove that $n^2 + n + 1$ is not prime. Since $n^2 + n + 1 > 1$, we'll need to prove the second part of the "or", i.e., that

$$\exists d \in \mathbb{N}, \ d \mid (n^2 + n + 1) \Rightarrow d = 1 \lor d = n^2 + n + 1.$$  

Let $d = \_\_\_\_$. We want to prove that $d \mid (n^2 + n + 1) \Rightarrow d = 1 \lor d = n^2 + n + 1$.

Then since $d \nmid (n^2 + n + 1)$, the implication is vacuously true.

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**Question 5**

Recall this theorem and these definitions from pg. 50 of the Course Notes:

- **Theorem 2.1. (Quotient-Remainder Theorem)** For all $n \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$, there exist $q, r \in \mathbb{Z}$ such that $n = qd + r$ and $0 \leq r < d$. Moreover, these $q$ and $r$ are unique (they are determined entirely by the values of $n$ and $d$).

- **Definition 2.2.** Let $n$, $d$, $q$, $r$ be the variables in the previous theorem. We say that $q$ and $r$ are the **quotient** and **remainder**, respectively, when $n$ is divided by $d$.

Match the following pairs of $n$ and $d$ with their corresponding quotient and remainder.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$d$</th>
<th>Quotient</th>
<th>Remainder</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = -10$, $d = 3$</td>
<td></td>
<td>[ Choose ]</td>
<td></td>
</tr>
<tr>
<td>$n = 10$, $d = 3$</td>
<td></td>
<td>[ Choose ]</td>
<td></td>
</tr>
<tr>
<td>n = 3, d = 10</td>
<td>[ Choose ]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>n = -3, d = 10</td>
<td>[ Choose ]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>