Topics this week:

**Midterm on Monday** (See exam page on course website)
Hour 3 - binary representations (a proof by induction)
Hour 4 - Worksheets: representing natural / real numbers in different bases, converting between bases, geometric series

**CSC165 Student Questions / Announcements**
- Problem Set 3 will be released at some point during reading week
- No lectures during reading week.
- There will be some office hours during reading week (check course website)
- Midterm Grades should be released tomorrow. Problem Set 2 grades soon as well.

Go to [menti.com](http://menti.com) and enter the code 17 27 04 (For Q+A during lecture)

**CSC 165 Lecture 11**

Binary Representation of Natural Numbers

Quick review of binary representation from last class:

\[ m \in \mathbb{N} \Rightarrow m = \sum_{i=0}^{k-1} b_i \times 2^i \]

**Notation**: \((b_{k-1} \ldots b_1 b_0)_2\)
We’ll go through a proof similar to Theorem 4.1 in the course notes (page 79).

\[ B(n, x) : \exists b_0, b_1, \ldots, b_{n-1} \in \{0, 1\}, \quad x = \sum_{i=0}^{n-1} b_i x^i, \]

where \( n \in \mathbb{Z}^+ \)
and \( x \in \mathbb{N} \).

Prove that:

\[ \forall n \in \mathbb{Z}^+, \forall x \in \mathbb{N}, x \leq 2^n - 1 \implies B(n, x). \]

Proof (by induction):

Base Case: Let \( n = 1 \). We’ll prove that

\[ \forall x \in \mathbb{N}, x \leq 2^1 - 1 \implies B(1, x) \]

Let \( x \in \mathbb{N} \). Assume \( x \leq 1 \) and show

\[ B(1, x) \]
Case 1: Assume that $x=0$. We will show that $B(1,0)$ is true.

Let $b_0 = 0$.  
$x = 0 = \sum_{i=0}^{0} b_i x^i = 0 \times 2^0 = 0$

Case 2: Assume $x \leq 1$.......

... Let $b_1 = 1$...

Induction Step: Let $k \in \mathbb{Z}^+$. Assume that

\[ \forall x \in \mathbb{N}, x \leq 2^k - 1 \Rightarrow B(k, x) \]

Want to prove that:

\[ \forall x \in \mathbb{N}, x \leq 2^{k+1} - 1 \Rightarrow B(k+1, x) \]

Let $x \in \mathbb{N}$. Assume $x \leq 2^{k+1} - 1$. And show $B(k+1, x)$.

Case 2: Assume $x > 2^k - 1$.  

Rough Work:

\[
\begin{array}{c}
\begin{array}{c}
\text{Case 2} \\
0 \\
I.P.
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
2^k \quad 2^{k+1} - 1
\end{array}
\end{array}
\]
Show $Q S x - 2^k \leq 2^{k-1}$

From A1:
- $x > 2^{k-1}$
- $x - 2^k > -1$
- $x - 2^k \geq 0$

From A2: $x \leq 2^{k+1} - 1$

$x - 2^k \leq 2^{k+1} - 2^k - 1$
$x - 2^k \leq 2^k (2-1) - 1$
$x - 2^k \leq 2^{k-1}$

From I.M. and the above, we know that $P(k, x - 2^k)$.

This means:
- $\exists b_0, b_1, ..., b_{k+1} \in \{0, 1\}$

Such that...

$x = \sum_{i=0}^{k} b_i \times 2^i + b_{k+1} \times 2^{k+1}$

$x - b_{k+1} \times 2^{k+1} = \sum_{i=0}^{k-1} b_i \times 2^i$

$\theta(k, x)$
\[ x - 2^k = \sum_{i=0}^{k-1} b_i \cdot 2^i \]

Let \( b_k = 1 \).

\[ x - 2^k = \sum_{i=0}^{k-1} b_i \cdot 2^i + b_k \cdot 2^k \]

\[ x = \sum_{i=0}^{k} b_i \cdot 2^i \]

By \( b_k = 1 \)

\[ \delta \cdot \theta(k < 1, x) \text{ is true} \]

Comments:

\[ \forall n \in \mathbb{Z}^+, \forall x \in \mathbb{N}, x \leq 2^{n-1} \Rightarrow \theta(n, x) \]

\[ x = \sum_{i=0}^{k-1} b_i \cdot 2^i \leq \sum_{i=0}^{k-1} 2^i \leq 2^k - 1 \]

\[ x = \left( b_{k-1} \ldots b_1 b_0 \right)_2 \times 2 \]