Intuition: how few edges can remain in a connected graph?

- cycles
- trees

Tree = connected but acyclic (no cycle)

A- For any graph $G = (V,E)$ that is connected,
   
   (*) $G$ is acyclic $\Rightarrow$ removing any edge from $G$ disconnects it

Proof: Let $G = (V,E)$ and assume $G$ is connected.
   
   We prove (*) by contrapositive.
   
   Assume $G$ contains some edge $(u,v)$ whose removal does not disconnect $G$.

   So $G$ contains a path from $u$ to $v$ that does not include edge $(u,v)$ — adding $(u,v)$ to this path creates a cycle in $G$.

Consequence: every connected graph on $n$ vertices contains at least as many edges as a tree on $n$ vertices.
B - Prove that for all trees, $|E| = |V| - 1$.

**Proof by induction:**

\[ \forall n \in \mathbb{Z}^+, \forall G = (V, E), |V| = n \land G \text{ is a tree} \Rightarrow |E| = n - 1 \]

- **Base Case:** EXERCISE

- **Ind. Hyp.:** Let $k \in \mathbb{Z}^+$ and assume
  \[ \forall G = (V, E), |V| = k \land G \text{ is a tree} \Rightarrow |E| = k - 1 \]

- **Ind. Step:** WTS $\forall G = (V, E), |V| = k+1 \land G \text{ is a tree} \Rightarrow |E| = k$

Let $G = (V, E)$ and assume $|V| = k+1$ and $G$ is a tree.

**ROUGH WORK**

Just remove some arbitrary $N \in V$

\[ G:\]

\[ G-\text{N}: \]

Need to ensure we remove $N$ s.t. $G-\text{N}$ is still connected.

Idea: find $N$ of degree 1 → number of neighbours

- **Assumption (to be proved later):** $G$ must contain at least one vertex $N_0$ with degree 1.

Then, $G' = (V', E')$ with

\[ V' = V - \{N_0\} \]

\[ E' = E - \{(N_0, N)\} \]

(where $N_0$ is the one neighbour of $N_0$ in $G$)

satisfies $|V'| = k$, $G'$ is a tree (still connected)

By I.H., $|E'| = k - 1$, so $|E| = |E'| + 1 = k$. □
Proof of "assumption":

Every tree with $n \geq 2$ vertices contains at least one "leaf" (vertex of degree 1).

Proof: Let $u \in V$. In the tree, find a longest path starting at $u$, and let $v$ be the end vertex on that path. Then $v$ is a leaf — otherwise, either $G$ would contain a cycle (contradicts $G$ is a tree) or there would be a longer path from $u$. 

[Diagram of a tree with a path and a leaf marked]