Let \( x \in \mathbb{N} \). A binary representation of \( x \) is a number \( k \in \mathbb{Z}^+ \) and numbers \( b_0, b_1, \ldots, b_{k-1} \in \{0, 1\} \) such that:

\[
x = \sum_{i=0}^{k-1} b_i \cdot 2^i
\]

In this case, we write \( x = (b_{k-1} \ldots b_1 b_0)_2 \).

\[
26 = 16 + 8 + 2 = 2^4 + 2^3 + 2^1
\]

\[
11010_2
\]

\[
26 = (11010)_2 = (000011010)_2
\]

The main predicate \( B : \mathbb{Z}^+ \times \mathbb{N} \to \{T, F\} \),

\[
B(n, x): \text{"} x \text{ has a binary representation using } n \text{ bits} \text{"}
\]

We’ll prove:

\[
\forall n \in \mathbb{Z}^+, \forall x \in \mathbb{N}, x \leq 2^{n-1} \implies B(n, x)
\]
\[ \forall x \in \mathbb{N}, x \leq 2^n \implies B(n, x) \]

\[ P(n) \]

"Every nat. number that's \( \leq 2^n - 1 \) has a binary representation using \( n \) bits."

Proof (by induction on \( n \))

**Base Case:** Let \( n = 1 \). We'll prove \( P(1) \).

Let \( x \in \mathbb{N} \). Assume \( x \leq 2^1 - 1 = 1 \).

So then \( x = 0 \) or \( x = 1 \). We'll prove \( B(1, x) \).

**Case 1** Assume \( x = 0 \).

Then \( x = (0)_{2} \), so it has a bin rep. using 1 bit.

**Case 2** Exercise :P

**Induction Step**

Let \( k \in \mathbb{N} \), and assume \( k \geq 1 \). Assume \( P(k) \), i.e., \( \forall x \in \mathbb{N}, x \leq 2^k - 1 \implies B(k, x) \).

We'll prove \( P(k+1) \), i.e., \( \forall x \in \mathbb{N}, x \leq 2^{k+1} - 1 \implies B(k+1, x) \).
Let \( x \in \mathbb{N} \). Assume \( x \leq 2^{k+1} - 1 \).
We'll prove that \( B(k+1, x) \) is True.

**Case 1** Assume \( x \geq 2^k \).
Let \( x' = x - 2^k \).
First,
\[
x' = x - 2^k \\
\leq 2^{k+1} - 1 - 2^k \\
= 2^k - 1 \quad (2^{k+1} = 2^k + 2^k)
\]
So by I.H. (subbing in \( x = x' \)), since \( x' \leq 2^k - 1 \), we conclude that \( x' \) has a bin rep using \( k \) bits, i.e., there exist \( b_0, b_1, \ldots, b_{k-1} \in \{0, 1\} \) that satisfy
\[
x' = \sum_{i=0}^{k-1} b_i \cdot 2^i.
\]
So then
\[
x' + 2^k = \sum_{i=0}^{k-1} b_i \cdot 2^i + 2^k \\
x = \sum_{i=0}^{k-1} b_i \cdot 2^i + 2^k
\]
\[ x = \sum_{i=0}^{k} b_i \cdot 2^i + 2^k. \]

So let \( b_k = 1 \), we get
\[ x = \sum_{i=0}^{k} b_i \cdot 2^i \]

Or in other words, \( x = (1 \ b_{k-1} \ldots \ b_1 \ b_0)_2 \), and so it has a bin. rep. using \( k+1 \) bits.

**Case 2** Assume \( x \leq 2^k - 1 \).

**Exercise:**

\[ \Box \]

**Remark:**

In fact, this is true:

\[ \forall n \in \mathbb{Z}^+, \forall x \in \mathbb{N}, \ x \leq 2^n - 1 \iff B(n,x) \]