Induction

Ex. Prove $\forall n \in \mathbb{N}, \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

Idea for induction

$P(n): \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$, where $n \in \mathbb{N}$

Want to prove $\forall n \in \mathbb{N}, P(n)$.

We'll prove two different things:

$\rightarrow 1$ $P(0)$ $\Rightarrow$ Base Case

$\rightarrow 2$ $\forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1)$

Induction Step

Proof Base Case

Part 1 Let $n=0$. Prove $P(0)$.
**Part 1** Let \( n = 0 \). Prove \( P(0) \), i.e.,
\[
\sum_{i=0}^{0} i = \frac{0(0+1)}{2}.
\]

**Exercise, just calculate**

**Part 2** Prove that \( \forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1) \).

Let \( k \in \mathbb{N} \).

Assume \( P(k) \), i.e.,
\[
\sum_{i=0}^{k} i = \frac{k(k+1)}{2}.
\]

We want to prove \( P(k+1) \), i.e.,
\[
\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}.
\]

We start with the LHS of the equation we want to prove:
\[
\sum_{i=0}^{k+1} i = \left( \sum_{i=0}^{k} i \right) + (k+1)
\]

By the I.H.,
\[
= \frac{k(k+1)}{2} + (k+1)
\]

(by our assumption)
\[
= \frac{k(k+1)+2(k+1)}{2}
\]

\[
= \frac{(k+1)(k+2)}{2}.
\]
Example

Prove that $\forall n \in \mathbb{N}, n \geq 3 \Rightarrow 2n+1 < 2^n$

Proof

Base Case: Let $n = 3$.

We'll prove that $2n+1 < 2^n$.

[exercise, calculate]

Induction step

We'll prove $\forall k \in \mathbb{N}, \ k \geq 3 \ W 2k+1 < 2^k \Rightarrow 2(k+1)+1 < 2^{k+1}$

Let $k \in \mathbb{N}$.

Assume $k \geq 3$ and that $2k+1 < 2^k$.

We'll prove that $2(k+1)+1 < 2^{k+1}$.
We start with the LHS of the desired inequality:
\[2(k+1) + 1 = 2k + 2 + 1 = 2k + 2\]
\[< 2^k + 2 \quad \text{(by I.H.)}\]
\[< 2^k + 2^k \quad \text{(since we assumed } k \geq 3)\]
\[= 2^{k+1}\]
\[< 2 \cdot 2^k \quad \text{(by I.H.)}\]
\[< 2^{k+1} \quad \text{(since } k > 1, \text{ we assumed } k \geq 3)\]
\[= 2^{k+1}\]