A proof is a mathematical argument that convinces someone else that a statement is true.

1. Explicit is better than implicit.
2. Rigid structure and language like programming.

Example 2.3

Prove that every real number $n$ greater than 20 satisfies the inequality $1.5n - 4 \geq 3$.

Translation

$$\forall n \in \mathbb{R}, \quad n > 20 \implies 1.5n - 4 \geq 3.$$

Proof.

**header:** Let $n \in \mathbb{R}$.

**Assume** that $n > 20$.

We want to prove that $1.5n - 4 \geq 3$.

**body:** We start with our assumption:

Discussion

(Rough work)

(Course Notes, manipulating inequalities)
then \( n > 20 \rightarrow 1.5n - 4 \geq 3 \) (since \( 26 \geq 3 \))

\[ n > 20 \rightarrow 1.5n - 4 \geq 3 \]

Discussion (Rough Work)

Proof Header

Proof Body

Then \( 1.5n - 4 \geq 3 \) (substituting 4)

\[ 1.5n - 4 \geq 3 \]

\[ 1.5n \geq 7 \]

\[ n \geq \frac{7}{1.5} \]

\[ n \geq 4.67 \]

Let \( n = 3 \)
Proof of 3. \( \text{Let } n = 3. \) \( \leq \) header

Then \( n > 20 \) is False, so \( n > 20 \implies 1.5n - 4 \geq 3 \) is vacuously True.

Proof of 4. \( \text{Let } n = 165. \) \( \leq \) header

We want to prove that \( n > 20 \) and \( 1.5n - 4 \geq 3. \)

Part 1: Prove \( n > 20. \)
This is true since \( 165 > 20. \)

Part 2: Prove that \( 1.5n - 4 \geq 3. \)
We calculate:
\[
1.5n - 4 = 1.5 (165) - 4 \\
= 247.5 - 4 \\
= 243.5 \\
\geq 3
\]

Important Reading
"What goes into a proof?"
What goes into a proof?
pp. 38-44

1. The proof header is important!
2. The statements in the proof body form a top-down chain of reasoning (implication).

Example

Prove that for all integers $x$, if $x$ divides $x+5$, then $x$ divides 5.

Translation

$\forall x \in \mathbb{Z}, \ x \mid x+5 \Rightarrow x \mid 5$

(exanding the definition of $\mid$)

$\forall x \in \mathbb{Z}, \ (\exists k, \in \mathbb{Z}, \ x+5=k \cdot x) \Rightarrow (\exists k \in \mathbb{Z}, \ 5=k \cdot x)$

Proof header

Let $x \in \mathbb{Z}$.

Assume there exists $k, \in \mathbb{Z}$ such that $x+5=k \cdot x$.

[meta: this introduces a new variable $k$,]
We want to prove that there exists an integer \( k_2 \), such that \( 5 = k_2 x \).

Let \( k_2 = \frac{k_1 - 1}{5} \).

We want to prove that \( 5 = k_2 x \).

**Rough work**

Goal: \( 5 = k_2 x \) \( \iff k_2 = ? \)

Assume: \( x + 5 = k_1 x \)

\[ \text{by definition of } k_2 \]

\[ k_2 = \frac{k_1 - 1}{5} \]

**Not always**

\[ k_2 = \frac{5}{x} \in \mathbb{Z} \]

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**Example (generalization)**

Original: “for all ints \( x \), if \( x \mid x + 5 \) then \( x \mid 5 \).”

Generalize: “for all ints \( d \) and \( x \), if \( x \mid x + d \) then \( x \mid d \).”

Translation: \( \forall d, x \in \mathbb{Z}, \; x \mid x + d \Rightarrow x \mid d \)

**Proof**

...
Let \( d \in \mathbb{Z} \)

Let \( x \in \mathbb{Z} \).

Assume there exists \( k, \in \mathbb{Z} \) such that
\[
x + d = k \cdot x.
\]

[meta: this introduces a new variable \( k \).]

We want to prove that
\[
\exists k_2 \in \mathbb{Z}, \ d = k_2 \cdot x.
\]

Let \( k_2 = \boxed{k - 1} \).

We want to prove that
\[
d = k_2 \cdot x.
\]

Let \( k_1 = \boxed{d \cdot x} \).

\[
d = k_2 \cdot x.
\]

\[
\text{Rough work}
\]

\[
\text{Goal: } d = k_2 \cdot x \quad \text{ } k_2 \text{ ?}
\]

\[
\text{Assume: } x + d = k_1 \cdot x
\]

\[
( k_2 = k_1 - 1 ) \quad \text{not always}
\]

\[
k_2 \neq \frac{d}{x} \notin \mathbb{Z}
\]

\[
\text{Homework!}
\]
For all ints $a, d, and x$, if $x | a x + d$, then $x | d$.\[ \]