A proof is a mathematical argument that convinces someone else that a statement is True.

Two principles for "intro" proofs
1. Proofs have a rigid structure and language (like programming).
2. Explicit is better than implicit.

Example 2.3

Prove that every real number $n$ greater than 20 satisfies the inequality $1.5n - 4 \geq 3$.

Translation

$\forall n \in \mathbb{R}, n > 20 \Rightarrow 1.5n - 4 \geq 3$.

Proof

header: Let $n \in \mathbb{R}$.

($n$ is an arbitrary $n > 20$)

Rough work (Discussion)

$n > 20$
(Let \( n \) be an arbitrary real number.)

**Assume** \( n > 20 \).

**We want to prove that**

\[ 1.5n - 4 \geq 3. \]

**Body:**

We start with our assumption:

\[ n > 20 \]

\[ \implies 1.5n > 30 \quad \text{(multiply by 1.5)} \]

\[ \implies 1.5n - 4 > 26 \quad \text{(subtract 4)} \]

\[ \implies 1.5n - 4 > 3 \quad \text{(since 26 > 3)} \]

\[ \implies 1.5n - 4 \geq 3 \quad (x \geq y \text{ means } x > y \lor x = y) \]

**Proof workflow**

**English**

\[ 1.5n - 4 > 3 \]

\[ 1.5n - 4 > 26 \]
\( \forall n \in \mathbb{R}, \, n > 20 \Rightarrow 1.5n - 4 \geq 3 \)

2. \( \forall n \in \mathbb{R}, \, n > 20 \land 1.5n - 4 \geq 3 \)

3. \( \exists n \in \mathbb{R}, \, n > 20 \Rightarrow 1.5n - 4 \geq 3 \leftarrow \)

4. \( \exists n \in \mathbb{R}, \, n > 20 \land 1.5n - 4 \geq 3 \)

Proof of 3

header: Let \( n = 1 \).

We'll prove that \( n > 20 \Rightarrow 1.5n - 4 \geq 3 \).

body: In this case, \( n > 20 \) is False.
So \( n > 20 \Rightarrow 1.5n - 4 \geq 3 \) is vacuously true.

---

**Important Reading!**

"What goes into a proof?" pp. 38-44

---

Example for

Prove that every integer \( x \), if \( x \) divides \( x + 5 \), then \( x \) divides 5.

Translation

\[
\forall x \in \mathbb{Z}, \ x \mid x + 5 \Rightarrow x \mid 5
\]

Or, expanding the definition of divisibility:

\[
\forall x \in \mathbb{Z}, \ (\exists k \in \mathbb{Z}, x + 5 = k_1x) \Rightarrow (\exists k_2 \in \mathbb{Z}, 5 = k_2x)
\]

Proof:

Let \( x \in \mathbb{Z} \).

\[
\begin{align*}
\text{Rough work} & \\
& x + 5 = k_1x
\end{align*}
\]
Assume there exists a \( k_1 \in \mathbb{Z} \) that satisfies
\[
x + 5 = k_1x.
\]
We want to prove that
\[
\exists k_2 \in \mathbb{Z}, \ 5 = k_2x.
\]
Let \( k_2 = \frac{k_1 - 1}{x} \).
We want to prove that \( 5 = k_2x \).

We start with our assumed equation:
\[
x + 5 = k_1x
\]
\[
5 = k_1x - x
\]
\[
5 = (k_1 - 1)x
\]
\[
5 = k_2x \quad \text{(since } k_2 = k_1 - 1\text{)}
\]

Generalization (example)

Example for every integer \( d \), and

Prove that

\[ x + 5 = k_1x \]
\[
\downarrow
\]
\[
k_2 = \frac{k_1 - 1}{x}
\]
\[
\downarrow
\]
\[
5 = k_2x
\]
\[
\times \frac{5}{x} \quad \text{not necessarily an int!}
\]
x divides x+d, then x divides d.

Translation
\[ \forall d \in \mathbb{Z}, \forall x \in \mathbb{Z}, x \mid x+d \Rightarrow x \mid d \]

Proof
Let \( d \in \mathbb{Z} \).
Let \( x \in \mathbb{Z} \).
Assume there exists a \( k_1 \in \mathbb{Z} \) that satisfies \( x + d = k_1x \).

We want to prove that \( \exists k_2 \in \mathbb{Z}, d = k_2x \).

Let \( k_2 = \frac{k_1 - 1}{x} \).

We want to prove that \( d = k_2x \).

We start with our assumed equation:
\[ x + d = k_1x \]
\[ d = k_1x - x \]
\[ d = (k_1 - 1) \times \]
\[ d = k_2 \times \text{ (since } k_2 = k_1 - 1) \]

Exercise example:

\[ \forall a \in \mathbb{Z}, \forall d \in \mathbb{Z}, \forall x \in \mathbb{Z}, \ x \mid ax + d \Rightarrow x \mid d \]