Last week: \( \forall G = (V, E), |E| \geq \frac{(|V|-1)(|V|-2)}{2} + 1 \Rightarrow G \) is connected

Today: \( \forall G = (V, E), |E| \leq \underline{\text{_______}} \Rightarrow G \) is not connected

Idea
1. Start with a connected graph \( G \).
2. Remove an edge but keep \( G \) connected.

\( |V| = 5 \)
\( |E| = 8 \rightarrow 4 \)

Theorem
\( \forall G = (V, E), (\exists e \in E, G - e \text{ is connected}) \Rightarrow G \) has a cycle.

Proof
Let \( G = (V, E) \). Assume there exists \( e \in E \) such that \( G - e \) is connected.
Let \( G = (V,E) \). Assume \(|V| = n \geq 3\) such that \( G-e \) is connected.
We'll prove that \( G \) has a cycle.

Let \( u, v \) be the endpoints of \( e \).

By our assumption, there exists a path \( P \) between \( u \) and \( v \) in \( G-e \).

So then adding edge \( e \) to \( P \) creates a cycle in \( G \).

\[ \]

5) Only for \( |V| \geq 2 \). Add to Course Notes
If \( |V|<2 \), \( E \) empty, so vacuously true.

Look at the contrapositive:

\( \forall G=(V,E), \ G \) does not have a cycle \( \Rightarrow \neg (\exists e \in E, \ G-e \text{ is connected}) \)

Idea
1) Start with a connected graph \( G \).
② Remove an edge but keep G connected.

③ Repeat until no more cycles.⇒ until G is a tree

Recall: a tree is a connected graph w/ no cycles.

Theorem
\[ \forall G = (V, E), \ G \ is \ a \ tree \Rightarrow |E| = |V| - 1 \]

Induction
\[ \forall n \in \mathbb{N}, \ n \geq 2 \Rightarrow (\forall G = (V, E), \ |V| = n \Rightarrow \begin{cases} (G \ is \ a \ tree) \\
|E| = |V| - 1 \end{cases}) \]

Proof
Proof

Base case \((n=2)\) see course notes

Induction step Let \(k \in \mathbb{N}\). Assume \(k \geq 2\).
Assume \(P(k)\). We’ll prove \(P(k+1)\).

Let \(G = (V, E)\). Assume \(|V| = k+1\).
Assume \(G\) is a tree.
We’ll prove that \(|E| = |V| - 1 = k\).

We’ll remove from \(G\) a leaf (vertex w/ one neighbour)
to obtain a new graph \(G' = (V', E')\).

External claim to prove

Every tree w/ \(|V| \geq 2\) has at least one leaf.

We know \(|V'| = k\). (Since we removed one
vertex).

Claim

\[ \text{G}' \text{ is tree.} \]

So by the IH,

\[ |E'| = |V'| - 1 \]

\[ |E'| + 1 = (|V'| + 1) - 1 \]

\[ |E| = |V| - 1 \]

(since we removed one edge and one vertex)

"Big idea" \cdot G connected??

\[ G \text{ must not be connected} \]

\[ G \text{ could be connected} \]

\[ G \text{ must be connected} \]
Proof by contradiction (last topic!)

Example For all graphs $G=(V,E)$, if $|V| \geq 2$
then there exist 2 different vertices
with the same degree.

$\Rightarrow$ # of neighbours

Proof by contradiction

Assume the statement is false, i.e.,
there exists a graph $G=(V,E)$ with $|V| \geq 2$
where all vertices have a different degree.

Let $n=|V|$.

So by our assumption,
there are $n$ different degrees in the graph.

But for every vertex, its degree is $\geq 0$,
and $\leq n-1$. 

Prove a contradiction

$Q \land \neg Q$
So the different degrees are \( \{0, 1, 2, \ldots, n-1\} \).

So \( E \) contains exactly one edge such that \( \deg(v_1) = 0 \),
and \( E \) contains exactly one edge such that \( \deg(v_2) = n-1 \).

On one hand, since
\[ \deg(v_1) = 0, \quad \text{\( u_1 \) and \( u_2 \) are not adjacent.} \]

On the other hand, since
\[ \deg(v_2) = n-1, \quad \text{\( u_1 \) and \( u_2 \) are adjacent.} \]

This is a contradiction, and so the original assumption must be false
(so the original statement is true).

**Summary**

Want to prove \( P \).

Proof by contradiction.

Prove \( P \implies Q \).

Proof by contrapositive.
Assume \( \neg P \).

\[ \therefore \]

Prove \( Q \).

\[ \therefore \]

Prove \( \neg Q \).

Assume \( \neg Q \).

\[ \therefore \]

Prove \( \neg P \).

Proof by contrapositive hiding in contradiction

Prove \( P \Rightarrow Q \).

\[ \overline{\text{Proof}} \]

Assume \( P \).

\[ \underline{\text{Assume } \neg Q. \ (\text{for a contradiction})} \]

\[ \therefore \]

Prove \( \neg P \), which contradicts our original assumption. (So \( Q \) is true.)