Recall that $\text{func}$ be an alg, and $I_n$ the set of its inputs of size $n$.

$$\text{WC}_{\text{func}}(n) = \max \left\{ \text{runtime of } \text{func}(x) \mid x \in I_n \right\}$$

Sometimes worst-case is too pessimistic. We define the average-case runtime as

$$\text{Avg}_{\text{func}}(n) = \text{avg} \left\{ \text{runtime of } \text{func}(x) \mid x \in I_n \right\}$$

$$= \frac{\sum \text{runtime at } \text{func}(x)}{|I_n|}$$
Example

def search_one (lst: List[int]) -> bool:
    for num in lst:
        if num == 1:
            return True
    return False

For this example, we'll consider only the input:

\( I_n = \text{the permutations of the numbers} \{1, 2, \ldots, n\} \)

\( I_0 = \{[]\} \)

\( I_1 = \{[1]\} \)

\( I_2 = \{[1, 2], [2, 1]\} \)

\( I_3 = \{[1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]\} \)
Fact: \( \forall n \in \mathbb{N}, |I_n| = n! = \prod_{i=1}^{n} i \)

Analysis (average-case)

From the definition:

\[
\text{Avg}_{n \rightarrow \infty}(n) = \sum_{1^\text{st} \in I_n} \frac{\text{runtime of search-one(1st)}}{|I_n|}
\]

\[
= \sum_{1^\text{st} \in I_n} \frac{n!}{|I_n|}
\]

For any 1st with \(1^\text{st}[i] = 1\), there are \(i+1\) iterations of the loop, taking \(i+1\) steps total.

Rough:

```
5 1 2 3 4 2 iters
---
1 2 3 5 4 1 iter
---
4 3 2 5 1 5 iters
---
0 1 2 \[
\vdots
\]
```

\( \vdots \)
Key idea: group inputs in $I_n$ based on where the 1 is.

$$\text{Avg}_{g_0}(n) = \frac{\sum_{1 \leq i \leq n!} \text{r.t. of } \text{search-one}(1st)}{n!}$$

$$= \frac{1}{n!} \sum_{i=0}^{n-1} \left( \frac{\sum_{1 \leq j \leq n} \text{r.t. of } \text{search-one}(1st)}{1 \leq i + [j] \leq 1} \right)$$

$$= \frac{1}{n!} \sum_{i=0}^{n-1} \left( \sum_{1 \leq i \leq n} (i+1) \right)$$

Key idea #2: for each $i$, there are $(n-1)!$ input lists where 1 is at index $i$.

Continuing from above:

$$\text{Avg}_{g_0}(n) = \frac{1}{n!} \sum_{i=0}^{n-1} (i+1)(n-1)!$$
\[
\frac{(n-1)!}{n!} \sum_{i=0}^{n-1} (i+1)
\]

\(\vdash\) (course notes)

\[
\frac{n+1}{2} \in \Theta(n)
\]