Mathematical Preliminaries

A set is a collection of distinct objects, called its elements.

\{3, 1, 10\}

\{ 'hello', 'goodbye' \}

the set of people in this room

Number sets

\(\mathbb{N}\), natural numbers \(\{0, 1, 2, 3, \ldots\}\)

\(\mathbb{Z}\), integers \(\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}\)

\(\mathbb{Z}^+\), positive ints \(\{1, 2, 3, \ldots\}\)

\(\mathbb{Q}\), rationals

\(\left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}\)

what the elements look like

variables and conditions
like

\[ \mathbb{R}, \text{ real numbers} \]

The empty set is \( \emptyset \) or \( \phi \).

The size of a set is its \# of elements.

\( |A| \) denotes the size of \( A \).

Set operations

\( \cdot x \in A: \text{"}x \text{ is an element of } A\text{"} \)

\( \cdot B \subseteq A: \text{"}B \text{ is a subset of } A\text{"} \)

\( A \subseteq A \rightarrow \text{always True} \)

\( \emptyset \subseteq A \rightarrow \text{always True} \)

\( \cdot A \cup B \text{ "union"} \)

\[ \{1, 2, 3, 4, 5\} \]

\( A \cap B \text{ "intersection"} \)

\[ \{2, 3\} \]

\( A = \{1, 2, 3\} \)

\( B = \{2, 4\} \)
\( A \setminus B \) "difference"
\( \emptyset, 3 \)  
\( A \times B \) "Cartesian product"
\( \{(1,2), (1,4), (2,2), (2,4), (3,2), (3,4)\} \)
\( A \times B = \{(p,q) \mid p \in A, q \in B\} \)
\( A \times \emptyset = \emptyset \)
\( (1,), (2,), (3,) \)

**Functions**

A function \( f: A \to B \) is a mapping from set \( A \) to set \( B \).

\( A \) is called the **domain** of \( f \).
\( B \) is called the **codomain** of \( f \).

\[ f: \mathbb{R} \to \mathbb{R} \]
\[ f(x) = x^2 - 5 \]
\[ \mathbb{R} \setminus \mathbb{N} \to \mathbb{R} \]
\[ f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \]
\[ f(x) = \frac{1}{x} \]

For \( k \in \mathbb{Z}^+ \), we say \( f \) is a \( k \)-ary function when it has the form

\[ f : A_1 \times A_2 \times \cdots \times A_k \to B \]

where each \( A_i \) is the set for one of the arguments.

Let \( f \) be a function. We say \( f \) is a predicate when its codomain is \( \{\text{True, False}\} \). Let \( \epsilon, \leq, =, \leq, \geq \).

**Summation and product notation**

\[
\frac{1+1^2}{3+1} + \frac{2+2^2}{3+2} + \cdots + \frac{100+100^2}{3+100}
\]

\[
\sum_{i=1}^{100} i + i^2
\]
$$= \sum_{i=1}^{100} \frac{i+i^2}{3+i} \cdot \prod_{i=1}^{100} \frac{i+i^2}{3+i} = \left(\frac{1+1^2}{3+1}\right)^x \left(\frac{2+2^2}{3+2}\right)^x \ldots \left(\frac{100+100^2}{3+100}\right)$$

$$\sum_{i=n}^{m} \frac{i+i^2}{3+i} \quad \text{What if } n > m? \quad \text{We call this an empty sum}$$

This is defined to have value 0.  

For an empty product, we give it the value 1.

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**Propositional Logic**

A proposition is a statement that is True or False.

- 3 < 4
- David is cool.
Python's list sort method is correct on all lists.

Propositional operators

- **NOT** (negation), symbol \( \neg \)
  - Truth table for **NOT**
    - \( p \) \( \neg p \)
    - False | True
    - True | False

- **AND** (conjunction), symbol \( \land \)
  - Truth table for **AND**
    - \( p \) \( q \) \( p \land q \)
    - False | False | False
    - False | True | False
    - True | False | False
    - True | True | True

- **OR** (disjunction), symbol \( \lor \)
  - Truth table for **OR**
    - \( p \) \( q \) \( p \lor q \)
    - False | False | False
    - False | True | False
    - True | False | True
    - True | True | True
\[
\begin{array}{c|c|c|c}
T & F & T \\
T & T & T \\
\end{array}
\]

"inclusive \(\Rightarrow\) 

or

- implies (implication, conditional), symbol \(\Rightarrow\)

- \(p \Rightarrow q\) is defined as \(\neg p \lor q\).

- Intuitively, "If \(p\) is True, then \(q\) is also True." 

\[
\begin{array}{c|c|c|c}
p & q & p \Rightarrow q \\
\hline
F & F & T \\
F & T & T \\
T & F & F \\
T & T & T \\
\end{array}
\]

For \(p \Rightarrow q\),

- we call \(p\) the hypothesis
- """ \(q\) the conclusion

we call \(q \Rightarrow p\) its converse

\textbf{NOT equivalent to } \(p \Rightarrow q\)!
(exercise: write truth table)

we call \( \neg q \Rightarrow \neg p \) its **contrapositive**. This is equivalent to \( p \Rightarrow q \)!

"If David is funny, then his students are laughing."

"If his students are not laughing, then David is not funny."

- **bi-implication/bi-conditional**, symbol \( \iff \).

  \( p \iff q \) is defined as \( (p \Rightarrow q) \land (q \Rightarrow p) \)
  
  or, as \( (p \land q) \lor (\neg p \land \neg q) \)
  
  "\( p \) if and only if \( q \)"

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