CSC165H1: Problem Set 3 Sample Solutions

Due Thursday March 7 2019, before 4pm

Note: solutions may be incomplete, and meant to be used as guidelines only. We encourage you to ask follow-up questions on the course forum or during office hours.

1. [7 marks] Induction and sequences. Consider the following definition of a sequence of numbers:

\[ d_n = \begin{cases} 
1 & \text{if } n = 0 \\
\frac{n}{d_{n-1}} & \text{otherwise}
\end{cases} \]

For example, \( d_0 = 1, d_1 = 1, d_2 = 2, d_3 = \frac{3}{2} \), and \( d_4 = \frac{8}{3} \). Our goal is to prove the following statement: “For all natural numbers \( n \), \( d_n > \sqrt{n} \) if and only if \( n \) is even.”

Complete the following two proofs, which together will prove the above statement.

(a) [4 marks] Prove the following using induction: \( \forall n \in \mathbb{Z}^+, d_{2n-1} \leq \sqrt{2n-1} \).

Hints: in the induction step, write \( d_{2k+1} \) in terms of \( d_{2k-1} \) using the given definition of the sequence. Later as an intermediate step, use difference of squares: \( (2k-1)(2k+1) = 4k^2 - 1 \).

Solution

Proof. Let \( P(n) \) be the predicate “\( d_{2n-1} \leq \sqrt{2n-1} \),” where \( n \in \mathbb{Z}^+ \). We want to prove the statement \( \forall n \in \mathbb{Z}^+, P(n) \), and will do so by induction.

Base case: Let \( n = 1 \). We want to show that \( d_1 \leq \sqrt{1} \). Because \( d_1 = 1 \), we know that \( d_1 \leq 1 = \sqrt{1} \) as desired.

Induction step: Let \( k \in \mathbb{Z}^+ \). We want to prove that \( P(k) \Rightarrow P(k+1) \). Assume \( P(k) \), i.e., that \( d_{2k-1} \leq \sqrt{2k-1} \). We want to show that \( d_{2k+1} \leq \sqrt{2k+1} \).

We have:

\[
\begin{align*}
    d_{2k+1} &= \frac{2k + 1}{d_2k} \\
         &= \frac{2k + 1}{\frac{2k}{d_{2k-1}}} \\
         &= \frac{2k + 1}{2k} \cdot d_{2k-1} \\
         &\leq \frac{2k + 1}{2k} \cdot \sqrt{2k - 1} \quad \text{(By the induction hypothesis)}
\end{align*}
\]

Page 1/9
At this point, we need to perform a calculation to show that \( \frac{2k + 1}{2k} \cdot \sqrt{2k - 1} \leq \sqrt{2k + 1} \).

\[
d_{2k+1} \leq \frac{2k + 1}{2k} \cdot \sqrt{2k - 1} \\
= \sqrt{2k + 1} \cdot \frac{2k + 1}{2k} \cdot \sqrt{2k - 1} \\
= \sqrt{2k + 1} \cdot \frac{4k^2 - 1}{2k} \\
< \sqrt{2k + 1} \cdot \frac{4k^2}{2k} \\
= \sqrt{2k + 1} \cdot \frac{2k}{2k} \\
= \sqrt{2k + 1}
\]

(b) [3 marks] Prove the following (with or without using induction): \( \forall n \in \mathbb{N}, d_{2n} > \sqrt{2n} \).
You may use the statement you proved in part (a) in this part.

Solution

Using the previous fact we actually don’t need induction! This question is a cool example of an “if and only if” proof where one of the implications can be used as an external fact in the proof of the other implication.

Proof. Let \( n \in \mathbb{N} \). We want to prove that \( d_{2n} > \sqrt{2n} \). We divide our proof into two cases.

Case 1: assume \( n = 0 \).
In this case, \( d_0 = 1 \) (by definition), and \( \sqrt{0} = 1 \), and so \( d_0 > \sqrt{0} \).

Case 2: assume \( n > 1 \).
Using the definition of the sequence \( d_n \):

\[
d_{2n} = \frac{2n}{d_{2n-1}} \\
\geq \frac{2n}{\sqrt{2n - 1}} \\
> \frac{2n}{\sqrt{2n}} \\
= \sqrt{2n}
\]

(by part (a), \( d_{2n-1} \leq \sqrt{2n - 1} \))

(since \( \sqrt{2n - 1} < \sqrt{2n} \))
2. **[9 marks] Number Representations.** As you might suspect, it is possible to represent numbers in other ways besides decimal (base-10) and binary (base-2). One intriguing representation is balanced ternary.

In balanced ternary, numbers are represented as sequences of digits \((d_{k-1}d_{k-2} \cdots d_1d_0)_{bt}\), where each digit \(d_i\) is \(T\), 0, or 1, with “\(T\)” used to represent the value \(-1\). The value of sequence \((d_{k-1}d_{k-2} \cdots d_1d_0)_{bt}\) is then simply \(\sum_{i=0}^{k-1} d_i \times 3^i\).

For example, \((T01)_{bt}\) represents the number \((-1) \times 3^2 + 0 \times 3^1 + 1 \times 3^0 = -9 + 1 = -8\).

(a) **[1 mark]**

(i) Write the decimal value of the balanced ternary number \((T011T)_{bt}\).

(ii) Write the balanced ternary representation of the decimal number 210 that doesn’t have any leading zeroes.

**Solution**

(i) \((T011T)_{bt} = (-1) \times 3^4 + 0 \times 3^3 + 1 \times 3^2 + 1 \times 3^1 + (-1) \times 3^0 = -81 + 9 + 3 = -70\)

(ii) \(210 = 243 + 0 \cdot 81 - 27 - 9 + 3 + 0 \cdot 3 = (10TT10)_{bt}\)

(b) **[3 marks]** Prove using induction that \(\forall n \in \mathbb{Z}^+, \ 6 \mid 3^n - 3\).

**Solution**

*Proof. Define* \(P(n) : 6 \mid 3^n - 3\), where \(n \in \mathbb{Z}^+.\) We prove \(\forall n \in \mathbb{Z}^+, \ 6 \mid 3^n - 3\) by induction.

**Base case.** \(P(1)\) is the statement that \(6 \mid 3^1 - 3\). Let \(k = 0\). Then, \(3^1 - 3 = 0 = 6 \cdot 0 = 6 \cdot k\) so \(\exists k \in \mathbb{Z}, 3^1 - 3 = 6k\).

**Inductive step:** Let \(k \in \mathbb{Z}^+.\) Assume \(6 \mid 3^k - 3\), i.e., that \(\exists d \in \mathbb{Z}, 3^k - 3 = 6d\). We’ll prove that \(6 \mid 3^{k+1} - 3\). Let \(d_1 = 3^{k-1} + d\).

Then

\[
3^{k+1} - 3 = 3 \cdot 3^k - 3 \\
= 2 \cdot 3^k + (3^k - 3) \\
= 6 \cdot 3^{k-1} + 6d \\
= 6d_1
\]

(by the I.H.)

\[\square\]

(c) **[5 marks]** Let \(x \in \mathbb{N}\) and \(n \in \mathbb{Z}^+.\) We say that \(x\) is \(n\)-digit positively balanced if and only if it has a balanced ternary representation \((d_{n-1}d_{n-2} \cdots d_1d_0)_{bt}\) in which none of the digits are equal to \(T\). For example, the decimal number 31 is 4-digit positively balanced, with the representation \((1011)_{bt}\), and it also is 6-digit positively balanced, with the representation \((001011)_{bt}\).

Prove, by induction, the following statement:

\[
\forall n \in \mathbb{Z}^+, \ \forall x \in \mathbb{N}, \ (x \text{ is } n\text{-digit positively balanced}) \Rightarrow 6 \nmid x - 2 \text{ and } 6 \nmid x - 5
\]

You may use part (b) and the Quotient-Remainder Theorem (QRT) in this question.

**Hint:** part of what the Quotient-Remainder Theorem says is that remainders are unique. So, for example, if you prove that \(6 \mid x - 4\), then \(x\) has remainder 4 when divided by 6, and so the QRT allows you to conclude that \(6 \nmid x - 0, 6 \nmid x - 1, 6 \nmid x - 2, 6 \nmid x - 3, \text{ and } 6 \nmid x - 5\).
Solution

Proof. Define $P(n)$: “$\forall x \in \mathbb{N}, \ (x \text{ is } n\text{-digit positively balanced}) \implies 6 \nmid x - 2 \land 6 \nmid x - 5$”, where $n \in \mathbb{N}$. We prove $\forall n \in \mathbb{Z}^+, \ P(n)$ by induction.

Base case: Let $n = 1$. We’ll prove $P(1)$.
Let $x \in \mathbb{N}$ and assume $x$ is 1-digit positively balanced, i.e., that there exists a digit $d_0$ such that $x = (d_0)_{bt}$ and $d_0$ is not $T$. The two possibilities are $d_0 = 0$ or $d_0 = 1$, and so $x = 0$ or $x = 1$.
If $x = 0$, we know from a calculation that $6 \nmid -2$ and $6 \nmid -5$, and so $6 \nmid x - 2 \land 6 \nmid x - 5$.
If $x = 1$, we again can calculate to see that $6 \nmid -1$ and $6 \nmid -4$, and so $6 \nmid x - 2 \land 6 \nmid x - 5$.

Induction step: Let $k \in \mathbb{Z}^+$ and assume that $\forall x \in \mathbb{N}, \ (x \text{ is } k\text{-digit positively balanced}) \implies 6 \nmid x - 2 \land 6 \nmid x - 5$. We’ll prove that $\forall x \in \mathbb{N}, \ (x \text{ is } (k+1)\text{-digit positively balanced}) \implies 6 \nmid x - 2 \land 6 \nmid x - 5$.
Let $x \in \mathbb{N}$ and assume that $x$ is $(k+1)$-digit positively balanced, i.e., that is has a $(k+1)$-digit balanced ternary representation $(d_kd_{k-1} \cdots d_1d_0)_{bt}$, where none of the digits are $T$. We want to prove that $6 \nmid x - 2 \land 6 \nmid x - 5$.
Let $x' = (d_{k-1} \cdots d_1d_0)_{bt}$. Then since there are $k$ digits, all of which are not $T$, we know that $x'$ is $k$-digit positively balanced. From this, we can use the induction hypothesis to conclude that $6 \nmid x' - 2$ and $6 \nmid x' - 5$.
Since we removed $d_k$ to obtain $x'$ from $x$, we know from the definition of balanced ternary representations that $x = x' + d_k \cdot 3^k$. There two possibilities for $d_k$: $0$ or $1$ (since we’ve assumed it can’t be $T$).

Case 1: assume $d_k = 0$.
In this case, $x = x'$, and so we’ve already proved that $6 \nmid x' - 2 \land 6 \nmid x' - 5$, we can conclude that $6 \nmid x - 2 \land 6 \nmid x - 5$.

Case 2: assume $d_k = 1$, so that $x = x' + 3^k$.
By the QRT and the fact that $6 \nmid x' - 2$ and $6 \nmid x' - 5$, we know that $x'$ has remainder 0, 1, 3, or 4 when divided by 6. That is, there exist $q_1, r_1 \in \mathbb{N}$ such that $x' = 6q_1 + r_1$, and $(r_1 = 0 \lor r_1 = 1 \lor r_1 = 3 \lor r_3 = 4)$.
Also, by part (b), we know that $6 \mid 3^k - 3$, i.e., there exists $q_2 \in \mathbb{Z}$ such that $3^k = 6q_2 + 3$.
The intuition here is that if you add the remainders from $x'$ and $3^k$, we can’t get 2 or 5. We’ll prove this formally using subcases on $r_1$.

Case 2(a): assume $r_1 = 0$.
Then, $x = x' + 3^k = (6q_1 + 0) + (6q_2 + 3) = 6(q_1 + q_2) + 3$, and so $x$ has remainder 3 when divided by 6. By the QRT, this means that $6 \nmid x - 2$ and $6 \nmid x - 5$.

Case 2(b): assume $r_1 = 1$.
Then, $x = x' + 3^k = (6q_1 + 1) + (6q_2 + 3) = 6(q_1 + q_2) + 4$, and so $x$ has remainder 4 when divided by 6. By the QRT, this means that $6 \nmid x - 2$ and $6 \nmid x - 5$. 
Case 2(c): assume $r_1 = 3$.
Then, $x = x' + 3^k = (6q_1 + 3) + (6q_2 + 3) = 6(q_1 + q_2 + 1) + 0$, and so $x$ has remainder 0 when divided by 6. By the QRT, this means that $6 \nmid x - 2$ and $6 \nmid x - 5$.

Case 2(d): assume $r_1 = 4$.
Then, $x = x' + 3^k = (6q_1 + 4) + (6q_2 + 3) = 6(q_1 + q_2 + 1) + 1$, and so $x$ has remainder 1 when divided by 6. By the QRT, this means that $6 \nmid x - 2$ and $6 \nmid x - 5$. 
3. [14 marks] **Properties of Asymptotic Notation.** Prove or disprove each of the following statements. You may use the “max”, ceiling, and floor functions in your solutions. However, you may not use any external facts of Big-Oh/Omega/Theta, and instead should only be using their definitions in your proofs.

The following definitions apply to part (d).

**Definition 1 (non-decreasing).** Let \( f : \mathbb{N} \to \mathbb{R}^+ \geq 0 \). We say that \( f \) is non-decreasing if and only if for all \( x, y \in \mathbb{N} \), if \( x \leq y \) then \( f(x) \leq f(y) \) (note that \( f(x) \) and \( f(y) \) can be equal).

**Definition 2 (power of two).** Let \( n \in \mathbb{N} \). We say that \( n \) is a power of two if and only if there exists a \( k \in \mathbb{N} \) such that \( n = 2^k \).

(a) [3 marks] \( \exists k \in \mathbb{N}, n^n \in \mathcal{O}(n^k) \). (Note: we define \( 0^0 = 0 \) for the purpose of this question.)

**Solution**

This statement is False, so we’ll disprove it by proving its negation:

\[
\forall k \in \mathbb{N}, \forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_0 \land n^n > c \cdot n^k
\]

**Proof.** Let \( k \in \mathbb{N} \), and \( c, n_0 \in \mathbb{R}^+ \). Let \( n = \lceil \max(n_0, c + 1, k + 1) \rceil \). (The ceiling is required to ensure \( n \in \mathbb{N} \).) We want to prove that \( n \geq n_0 \) and \( n^n > c \cdot n^k \).

**Part 1:** proving that \( n \geq n_0 \).

We can prove this directly from our definition of \( n \):

\[
n = \lceil \max(n_0, c, k + 1) \rceil \geq \max(n_0, c, k + 1) \geq n_0
\]

**Part 2:** proving that \( n^n > c \cdot n^k \).

By our definition of \( n \), we know that \( n \geq k + 1 \). From the latter inequality, we have:

\[
\begin{align*}
n &\geq k + 1 \\
n - 1 &\geq k \\
n^{n-1} &\geq n^k
\end{align*}
\]

(raising \( n \) to the power of both sides)

We also know from the definition of \( n \) that \( n \geq c + 1 \). So we can then multiply this with the above inequality to obtain

\[
\begin{align*}
n \cdot n^{n-1} &\geq (c + 1) \cdot n^k \\
n^n &\geq (c + 1) \cdot n^k \\
n^n &> c \cdot n^k
\end{align*}
\]

(b) [3 marks] \( 165n^5 + n^2 \in \mathcal{O}(n^5 - n^3) \).

**Solution**

This statement is True, so we’ll prove it.

**Proof.** Let \( c = 167 \) and \( n_0 = 20 \). Let \( n \in \mathbb{N} \), and assume that \( n \geq n_0 \). We’ll prove that
165n^5 + n^2 \leq c(n^5 - n^3).

First, we know that since \( n \geq 0 \), \( n^5 \geq n^2 \). We'll call this Inequality 1. Also, since we assumed \( n \geq n_0 \), we have:

\[
\begin{align*}
  n &\geq n_0 \\
  n &\geq 20 \quad \text{(by definition of } n_0) \\
  n^2 &\geq 400 \\
  n^5 &\geq 400n^3 \\
  n^5 &\geq 167n^3 \quad \text{(Inequality 2)}
\end{align*}
\]

Now, adding Inequalities 1 and 2 yields:

\[
\begin{align*}
  n^2 + 167n^3 &\leq 2n^5 \\
  165n^5 + n^2 + 167n^3 &\leq 167n^5 \\
  165n^5 + n^2 &\leq 167n^5 - 167n^3 \\
  165n^5 + n^2 &\leq 167(n^5 - n^3) \\
  165n^5 + n^2 &\leq c(n^5 - n^3) \quad \text{(by definition of } c) \\
\end{align*}
\]

(c) [4 marks] \( 4n^2 \in \Theta(4^{n^2 + n}) \).

**Solution**

This statement is False. We’ll disprove it by proving its negation:

\( 4n^2 \not\in O(4^{n^2 + n}) \lor 4n^2 \not\in \Omega(4^{n^2 + n}) \)

**Proof.** Since this negated statement is an OR, we can prove it by proving just one of the two parts. We’ll prove that \( 4n^2 \not\in \Omega(4^{n^2 + n}) \).

Expanding the negated definition of Omega, we need to prove that \( \forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_0 \land 4n^2 < c \cdot 4^{n^2 + n} \).

Let \( c, n_0 \in \mathbb{R}^+ \). Let \( n = \left\lceil \max\left(n_0, \log_4 \frac{1}{c} + 1\right) \right\rceil \). We need to prove that \( n \geq n_0 \) and that \( 4n^2 < c \cdot 4^{n^2 + n} \).

**Part 1**: proving that \( n \geq n_0 \).

By our definition of \( n \), we know that \( n \geq \max\left(n_0, \log_4 \frac{1}{c} + 1\right) \), and so \( n \geq n_0 \).

**Part 2**: proving that \( 4n^2 < c \cdot 4^{n^2 + n} \).

By our definition of \( n \), we know that \( n \geq \log_4 \frac{1}{c} + 1 \). We’ll start with the right-hand side of the
inequality we want to prove:
\[ c \cdot 4^{n^2+n} = c \cdot 4^{n^2} \cdot 4^n \]
\[ \geq c \cdot 4^{n^2} \cdot 4^{\log_4 \frac{1}{c} + 1} \quad \text{(since } n \geq \log_4 \frac{1}{c} + 1) \]
\[ = c \cdot 4^{n^2} \cdot \frac{1}{c} \cdot 4 \]
\[ = 4^{n^2} \cdot 4 \]
\[ > 4^{n^2} \quad \text{(since } 4 > 1) \]

"In fact, \(4^n\) is actually Big-Oh of \(4^{n^2+n}\), so it’s impossible to prove \(4^n \notin O(4^{n^2+n})\)!

\[ \square \]

(d) [4 marks] For every function \(f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}\), if \(f\) is non-decreasing and \(f(n) = n^2\) for every \(n \in \mathbb{N}\) that is a power of two, then \(f \in \Theta(n^2)\).

You may use these facts about ceiling and floor:
\[ \forall x \in \mathbb{R}, \exists \varepsilon \in \mathbb{R}, \lceil x \rceil = x + \varepsilon \wedge 0 \leq \varepsilon < 1 \quad \text{(Fact 1)} \]
\[ \forall x \in \mathbb{R}, \exists \varepsilon \in \mathbb{R}, \lfloor x \rfloor = x - \varepsilon \wedge 0 \leq \varepsilon < 1 \quad \text{(Fact 2)} \]

**Solution**

We will prove the statement.

*Proof.* Let \(f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}\), and assume that \(f\) is non-decreasing, and that \(f(n) = n^2\) for every power of two \(n \in \mathbb{N}\). Let \(c_1 = 1/4\), \(c_2 = 4\), and \(n_0 = 1\). Let \(n \in \mathbb{N}\), and assume \(n \geq n_0\). We’ll prove that \(c_1 n^2 \leq f(n) \leq c_2 n^2\). Since this is actually an AND, we’ll prove the two inequalities separately.

**Part 1:** proving that \(c_1 n^2 \leq f(n)\).

Let \(k_1 = \lceil \log n \rceil\). Let \(\varepsilon_1 = \log n - k_1\); by Fact 2, we know that \(0 \leq \varepsilon_1 < 1\). Since \(k_1 \leq \log n\), we know \(2^{k_1} \leq 2^{\log n} = n\). So by the assumption that \(f\) is non-decreasing, we know that \(f(2^{k_1}) \leq f(n)\). So we need to now prove that \(f(2^{k_1}) \geq c_1 n^2\).

\[ f(n) \geq f(2^{k_1}) \]
\[ = (2^{k_1})^2 \quad \text{(by second assumption on } f) \]
\[ = 4^{k_1} \]
\[ = 4^{\log n - \varepsilon_1} \]
\[ = \frac{n^2}{4^{\varepsilon_1}} \]
\[ > \frac{n^2}{4} \quad \text{(since } \varepsilon_1 < 1) \]
\[ = c_1 n^2 \quad \text{(by the definition of } c_1) \]

**Part 2:** proving that \(f(n) \leq c_2 n^2\).

Let \(k_2 = \lceil \log n \rceil\). Let \(\varepsilon_2 = k_2 - \log n\); by Fact 1, we know that \(0 \leq \varepsilon_2 < 1\). Since \(k_2 \geq \log n\), and the assumption that \(f\) is non-decreasing, we know that \(f(2^{k_2}) \geq f(n)\). So we need to now
prove that \( f(2^{k_2}) \leq c_2 n^2 \).

\[
\begin{align*}
  f(n) &\leq f(2^{k_2}) \\
  &= (2^{k_2})^2 \\
  &= 4^{k_2} \\
  &= 4^\log n + \varepsilon_2 \\
  &= 4^{\varepsilon_2} \cdot n^2 \\
  &< 4n^2 \\
  &= c_2 n^2 
\end{align*}
\]

(by second assumption on \( f \))

(since \( \varepsilon_2 < 1 \))

(by the definition of \( c_2 \))