Take a deep breath.
This is your chance to show us
How much you’ve learned.
We WANT to give you the credit
That you’ve earned.
A number does not define you.
Good luck!

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<tr>
<th>Question</th>
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Use this page for rough work. If you want work on this page to be marked, please indicate this clearly at the location of the original question.
1. [8 marks] Short answers questions.

(a) [2 marks] Let $U = \{a, b\}$. Let $S_1$ be the set of strings over $U$ that start and end with different letters, and let $S_2$ be the set of strings over $U$ with length 3. Write down all the elements of $S_2 \setminus S_1$.

**Solution**

$S_2 \setminus S_1 = \{aaa, aba, bab, bbb\}$

(b) [3 marks] Write down a truth table for the following expression in propositional logic. Rough work (e.g., intermediate columns of the truth table) is not required, but can be included if you want.

$$(p \Rightarrow q) \iff \neg r$$

**Solution**

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$r$</th>
<th>$(p \Rightarrow q) \iff \neg r$</th>
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(c) [3 marks] Consider the following statement (assume predicates $P$ and $Q$ have already been defined):

$$\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, \ P(x) \Rightarrow Q(x, y) \lor Q(x, y + 1)$$

Suppose we want to disprove this statement. Write the complete proof header for a disproof; you may write statements like “Let $x = _____” without filling in the blank. The last statement of your proof header should be “We will prove that…” where you clearly state what’s left to prove, in the same style as the lectures or the Course Notes.

You do not need to include any other work (but clearly mark any rough work you happen to use).

**Solution**

Let $x = _____. \ Let \ y \in \mathbb{N}. \ We \ will \ prove \ that \ P(x) \ is \ True \ and \ that \ Q(x, y) \ and \ Q(x, y + 1) \ are \ False.$
2. [7 marks] Translations. Let $T$ be the set of all hockey teams, and suppose we define the following predicates:

- $\text{Canadian}(x)$: “$x$ is a Canadian team”, where $x \in T$
- $\text{Star}(x)$: “$x$ has at least one player who is a superstar”, where $x \in T$
- $\text{Defeated}(x,y)$: “$x$ defeated $y$ at least once”, where $x,y \in T$ (note that $\text{Defeated}(x,y)$ does not mean the same thing as $\text{Defeated}(y,x)$)

Translate each of the following statements. No explanation is necessary. Do not define any of your own predicates or sets. You may use the = and $\neq$ symbols to compare whether two teams are the same.

(a) [1 mark] Every Canadian team has at least one superstar player.

Solution
$\forall x \in T, \text{Canadian}(x) \Rightarrow \text{Star}(x)$

(b) [2 marks] Every Canadian team has defeated every non-Canadian team.

Solution
$\forall x \in T, \text{Canadian}(x) \Rightarrow (\forall y \in T, \neg \text{Canadian}(y) \Rightarrow \text{Defeated}(x,y))$

Or equivalently:
$\forall x,y \in T, \text{Canadian}(x) \land \neg \text{Canadian}(y) \Rightarrow \text{Defeated}(x,y)$

(c) [2 marks] If at least one Canadian team has a superstar, then every Canadian team has defeated at least one other team. (Note: “other” means that the second team must be different from the first.)

Solution
$\left( \exists x \in T, \text{Canadian}(x) \land \text{Star}(x) \right) \Rightarrow \left( \forall x \in T, \text{Canadian}(x) \Rightarrow (\exists y \in T, y \neq x \land \text{Defeated}(x,y)) \right)$

(d) [2 marks] There is a Canadian team with a superstar, but every other Canadian team does not have a superstar. (Note: “other” means that the second team must be different from the first.)

Solution
$\exists x \in T, \text{Canadian}(x) \land \text{Star}(x) \land (\forall y \in T, y \neq x \land \text{Canadian}(x) \Rightarrow \neg \text{Star}(y))$
3. [6 marks] A proof about numbers. Consider the following statement about natural numbers: “Every odd natural number greater than one is a difference of squares.” (Recall that natural number \( n \) is odd when \( n = 2k + 1 \) for some \( k \in \mathbb{N} \), and \( n \) is a difference of squares when \( n = p^2 - q^2 \) for some \( p, q \in \mathbb{Z}^+ \).)

(a) [2 marks] Translate the above statement into predicate logic. Do not use the Odd or DifferenceOfSquares predicates in your answer (instead, use the definitions provided above).

**Solution**

\[
\forall n \in \mathbb{N}, n > 1 \land (\exists k \in \mathbb{N}, n = 2k + 1) \Rightarrow (\exists p, q \in \mathbb{Z}^+, n = p^2 - q^2)
\]

(b) [4 marks] Prove or disprove the above statement. If you choose to disprove the statement, you must start by writing its negation. We have left you space for rough work here and on the next page, but write your formal proof in the box below.

**Hint:** Remember that \( \forall m \in \mathbb{N}, (m + 1)^2 = m^2 + 2m + 1 \).

**Solution**

**Proof.** Let \( n \in \mathbb{N} \). Assume \( n > 1 \), and that \( \exists k \in \mathbb{N}, n = 2k + 1 \). (Since \( n > 1 \), we know that \( k > 0 \).)

Let \( p = k + 1 \) and \( q = k \).

Then, \( p^2 - q^2 = (k + 1)^2 - k^2 = (k^2 + 2k + 1) - k^2 = 2k + 1 = n \).

\( \square \)
Use this page for rough work. If you want work on this page to be marked, please indicate this clearly at the location of the original question.
4. [5 marks] Divisibility. Recall the following fact about divisibility.

\[ \forall d, n \in \mathbb{N}, \, d \mid n \Rightarrow 1 \leq d \leq n \]  

(Fact 1)

Use this fact to prove the following statement.

\[ \forall d, n \in \mathbb{N}, \, d \mid n \land d \neq n \Rightarrow d \leq n/2 \]

Clearly state where you use the fact in your proof. We have left you space for rough work here and on the next page, but write your formal proof in the box below.

Solution

Note: we made a correction on the test: in both of the above statements, the domain of the variables \( d \) and \( n \) should be \( \mathbb{Z}^+ \), not \( \mathbb{N} \).

Solution

Proof. Let \( d, n \in \mathbb{Z}^+ \). Assume that \( d \mid n \), i.e., that there exists a \( k \in \mathbb{Z} \) such that \( n = kd \), and assume that \( d \neq n \).

By Fact 1, since \( d \mid n \) we know that \( 1 \leq d \leq n \). Since we assumed that \( d \neq n \), we can conclude that \( d < n \). Then since \( n = kd \) and \( d < n \), we know that \( k > 1 \). And since \( k \) is an integer, we conclude that \( k \geq 2 \).

So then multiplying both sides by \( d \), we get:

\[
\begin{align*}
    k & \geq 2 \\
    kd & \geq 2d \\
    n & \geq 2d \\
    d & \leq \frac{n}{2} 
\end{align*}
\]

(since \( n = kd \))

\( \square \)
Use this page for rough work. If you want work on this page to be marked, please indicate this clearly at the location of the original question.