Name:
Student Number:
UTORid:

Please read the following guidelines carefully!

• Please print your name, UTORid and student number above.
• This test has 5 questions. There are a total of 10 pages, DOUBLE-SIDED.
• Answer questions clearly and completely. Provide justification unless explicitly asked not to.
• Unless stated otherwise, your formulas can use only the propositional connectives and quantifiers we have seen in class, arithmetic operators (like +, ×, and exponentiation), comparison operators (like =, > and ≥), and the divisibility and Prime predicates. You may not define your own sets or predicates unless asked to do so.
• All formulas must have negations applied directly to propositional variables or predicates.
• Formal proofs should follow the guidelines used in the course (e.g., explicitly introduce all variables and assumptions, clearly state all assumptions and reasoning you make in your proof body, etc.) In your proofs, you may use definitions of predicates from the course.
• You may not use induction for your proofs on this test.

<table>
<thead>
<tr>
<th>Question</th>
<th>Grade</th>
<th>Out of</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1</td>
<td>5</td>
<td></td>
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<tr>
<td>Q2</td>
<td>5</td>
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<tr>
<td>Q3</td>
<td>6</td>
<td></td>
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<tr>
<td>Q4</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Q5</td>
<td>9</td>
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<tr>
<td>Total</td>
<td>30</td>
<td></td>
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On this test you may use the following standard equivalences.

**Standard Equivalences** (where \( P, Q, P(x), Q(x), \text{ etc. } \) are arbitrary sentences)

- **Commutativity**
  \[ P \land Q \iff Q \land P \]
  \[ P \lor Q \iff Q \lor P \]
  \[ P \iff Q \iff Q \iff P \]

- **Associativity**
  \[ P \land (Q \land R) \iff (P \land Q) \land R \]
  \[ P \lor (Q \lor R) \iff (P \lor Q) \lor R \]

- **Identity**
  \[ P \land (Q \lor \neg Q) \iff P \]
  \[ P \lor (Q \land \neg Q) \iff P \]

- **Absorption**
  \[ P \land (Q \land \neg Q) \iff Q \land \neg Q \]
  \[ P \lor (Q \lor \neg Q) \iff Q \lor \neg Q \]

- **Idempotency**
  \[ P \land P \iff P \]
  \[ P \lor P \iff P \]

- **Double Negation**
  \[ \neg \neg P \iff P \]

- **DeMorgan’s Laws**
  \[ \neg (P \land Q) \iff \neg P \lor \neg Q \]
  \[ \neg (P \lor Q) \iff \neg P \land \neg Q \]

- **Distributivity**
  \[ P \land (Q \lor R) \iff (P \land Q) \lor (P \land R) \]
  \[ P \lor (Q \land R) \iff (P \lor Q) \land (P \lor R) \]

- **Contrapositive**
  \[ P \Rightarrow Q \iff \neg Q \Rightarrow \neg P \]

- **Implication**
  \[ P \Rightarrow Q \iff \neg P \lor Q \]

- **Biconditional**
  \[ P \iff Q \iff (P \Rightarrow Q) \land (Q \Rightarrow P) \]

- **Renaming**
  \[ \text{(where } P(x) \text{ does not contain variable } y) \]
  \[ \forall x, P(x) \iff \forall y, P(y) \]
  \[ \exists x, P(x) \iff \exists y, P(y) \]

- **Quantifier Negation**
  \[ \neg \forall x, P(x) \iff \exists x, \neg P(x) \]
  \[ \neg \exists x, P(x) \iff \forall x, \neg P(x) \]

- **Quantifier Commutativity**
  \[ \forall x, \forall y, S(x, y) \iff \forall y, \forall x, S(x, y) \]
  \[ \exists x, \exists y, S(x, y) \iff \exists y, \exists x, S(x, y) \]

- **Quantifier Distributivity**
  \[ \text{(where } S \text{ does not contain variable } x) \]
  \[ S \land \forall x, Q(x) \iff \forall x, S \land Q(x) \]
  \[ S \lor \forall x, Q(x) \iff \forall x, S \lor Q(x) \]
  \[ S \land \exists x, Q(x) \iff \exists x, S \land Q(x) \]
  \[ S \lor \exists x, Q(x) \iff \exists x, S \lor Q(x) \]
  \[ (\forall x, P(x)) \land (\forall x, Q(x)) \iff \forall x, P(x) \land Q(x) \]
  \[ (\exists x, P(x)) \lor (\exists x, Q(x)) \iff \exists x, P(x) \lor Q(x) \]
1. [5 marks] Short answer questions.

(a) [1 mark] Give example values of the propositional variables \( p, q, r \) and \( s \) that make the following logical expression evaluate to True: \((p \land q) \Rightarrow (r \Rightarrow s)\)

(b) [1 mark] Write an expression that is logically equivalent to \( \neg a \land b \Rightarrow c \) and that has both of the variables \( a \) and \( c \) on the right hand side of an implication. Show your work. You are not required to state the names of the logical equivalence rules that you apply.

(c) [1 mark] Do the sets \( T_1 = \{ x \mid x \in \mathbb{Z} \text{ and } \exists k \in \mathbb{Z}, x = 2k \} \) and \( T_2 = \{ x \mid x \in \mathbb{Z} \text{ and } \exists k \in \mathbb{Z}, x = 2k - 1 \} \) give a partition of \( \mathbb{Z} \)? Explain.

(d) [1 mark] Consider the statement \( \forall x \in \mathbb{Z}, \exists y \in \mathbb{N}, x + y > 165 \) and the argument:

Let \( x = 0 \) and \( y = 166 \). Then \( x + y = 0 + 166 = 166 \)

This argument is not a correct proof of the statement. Explain why not.

(e) [1 mark] Compute the numerical value of \( \sum_{i=0}^{2} (-1)^i (2i) \). 

2. [5 marks] Propositional logic.

(a) [3 marks] Write the truth table for the following formula. No rough work is required.

\[(p \iff q) \Rightarrow r\]

(b) [2 marks] Write a formula that is logically equivalent to the formula from Part (a) that only uses the operators: \(\neg\), \(\land\), \(\lor\). Show your work. You do not need to give the names of any equivalence rules. You do not need to simplify your final answer.

Reminder: Review and follow the guidelines stated on Page 1 of this test.
3. [6 marks] **Translations.** Consider the domain \( D = \{ \text{all tests and students} \} \), and the predicate symbols \( T(x): \text{“} x \text{ is a test”} \), \( S(x): \text{“} x \text{ is a student”} \), \( L(x): \text{“} x \text{ attended the lecture”} \), \( C(x,y): \text{“} x \text{ found } y \text{ challenging”} \), and \( W(x,y): \text{“} x \text{ did well on } y \text{”} \).

Using only these symbols and appropriate connectives and quantifiers, translate each sentence below. That is, give a natural English sentence that corresponds to each given symbolic sentence, and give a clear symbolic sentence that corresponds to each given English sentence. Quantifiers may **only** be used on the domain \( D \).

(a) [1 mark] Some student did not attend the lecture.

(b) [1 mark] \( \forall x \in D, S(x) \Rightarrow L(x) \)

(c) [1 mark] Some student did well on some test.

(d) [1 mark] Students do not do well on tests.

(e) [2 marks] All students who attended the lecture did well on some test.
Consider the predicate \( M(n) : \exists k \in \mathbb{N}, n = 5k + 2 \), where \( n \in \mathbb{N} \).

(a) [1 mark] Give one example of a natural number \( n \) for which \( M(n) \) is True, and then show that \( M(2n + 3) \) is also True for your choice of \( n \). \textbf{HINT}: Try using \( k = 0 \) to determine your \( n \).

(b) [4 marks] Prove or disprove the statement: \( \forall n \in \mathbb{N}, M(n) \Rightarrow M(2n + 3) \).
If you choose to disprove the statement, you must start by writing its negation.
5. **[9 marks] Working with sets and operations.** Consider the set \( T = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\} \). Each element of \( T \) is a 2-tuple of real numbers. For example, \((1.0, 6.5) \in T\).

We can define the addition (+) and multiplication (\(*\)) operations on the set \( T \) as follows:

For \( x_1, x_2, y_1, y_2 \in \mathbb{R} \), we define

- \((x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)\)
- \((x_1, y_1) \ast (x_2, y_2) = (x_1 \ast x_2, y_1 \ast y_2)\)

For example, \((1, 4) + (2, 5) = (3, 9)\) and \((1, 4) \ast (2, 5) = (2, 20)\).

A set \( S \) is said to be **closed** under a given operation if performing that operation using elements of \( S \) always gives a result that is an element of \( S \). For example, \( \mathbb{N} \) is closed under addition, because the sum of any two natural number is itself a natural number. However, \( \mathbb{N} \) is not closed under subtraction, because, although \( 5 \) and \( 9 \) are both natural numbers, \( 5 - 9 = -4 \) is not.

Now consider the set \( U = \{(x, y) \mid (x, y) \in T \text{ and } y = x^2\} \), together with the operations + and \(*\), as defined above.

(a) **[1 mark]** Give one example of an element of \( U \).

(b) **[1 mark]** Translate the statement “The set \( U \) is closed under the operation +.” into predicate logic. Quantify over the set \( \mathbb{R} \).

(c) **[3 marks]** Prove or disprove the statement: “The set \( U \) is closed under the operation +.”

If you choose to disprove the statement, you must start by writing its negation.

\[\text{Discussion.}\]

\[\text{Proof.}\]
(d) **[1 mark]** Translate the statement “The set U is closed under the operation \(*\).” into predicate logic. Quantify over the set \(\mathbb{R}\).

(e) **[3 marks]** Prove or disprove the following statement: “The set U is closed under the operation \(*\).” If you choose to disprove the statement, you must start by writing its negation.

**Discussion.**

**Proof.**
Use this page for rough work. If you want work on this page to be marked, please indicate this clearly at the location of the original question.
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