Name: Sample Solutions
Student Number: Sample Solutions
UTORid: Sample Solutions

Please read the following guidelines carefully!

- Please print your name,UTORid and student number above.
- This test has 5 questions. There are a total of 10 pages, DOUBLE-SIDED.
- Answer questions clearly and completely. Provide justification unless explicitly asked not to.
- Unless stated otherwise, your formulas can use only the propositional connectives and quantifiers we have seen in class, arithmetic operators (like +, ×, and exponentiation), comparison operators (like =, > and ≥), and the divisibility and Prime predicates. You may not define your own sets or predicates unless asked to do so.
- All formulas must have negations applied directly to propositional variables or predicates.
- Formal proofs should follow the guidelines used in the course (e.g., explicitly introduce all variables and assumptions, clearly state all assumptions and reasoning you make in your proof body, etc.) In your proofs, you may use definitions of predicates from the course.
- You may not use induction for your proofs on this test.

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<th>Question</th>
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1. [5 marks] Short answer questions.

(a) [1 mark] Give example values of the propositional variables \( p, q, r \) and \( s \) that make the following logical expression evaluate to True: \(((p \land q) \Rightarrow r) \Rightarrow s\)

Choosing \( s = \text{True} \) makes the outer implication True with any settings of \( p, q, r \). So \( p = \text{True}, \ r = \text{True} \)

(b) [1 mark] Write an expression that is logically equivalent to \( a = \neg b \lor c \) and that has both of the variables \( a \) and \( b \) on the left hand side of an implication. Show your work. You are not required to state the names of the logical equivalence rules that you apply.

\[
\begin{align*}
a &\Rightarrow \neg b \lor c \\
\iff & \ a = (\neg b \lor c) \quad \text{by precedence rules} \\
\iff & \ \neg a \lor \neg b \lor c \\
\iff & \ \neg (a \land b) \lor c \\
\iff & \ a \land b \Rightarrow c
\end{align*}
\]

(c) [1 mark] Do the sets \( T_1 = \{x \mid x \in \mathbb{Z} \text{ and } \exists k \in \mathbb{Z}, x = 2k\} \) and \( T_2 = \{x \mid x \in \mathbb{Z} \text{ and } \exists k \in \mathbb{Z}, x = 3k\} \) give a partition of \( \mathbb{Z} \)? Explain.

No. \( T_1 \) is the set of all even integers and \( T_2 \) is the set of all multiples of \( 3 \). \( 6 \in T_1 \) and \( 6 \in T_2 \), so \( T_1 \cap T_2 \neq \emptyset \)

Also, \( 5 \notin \mathbb{Z} \) but \( 5 \notin T_1 \cup T_2 \).

(d) [1 mark] Consider the statement \( \exists x \in \mathbb{Z}, \forall y \in \mathbb{N}, x + y > 165 \) and the argument:

Let \( x = 166 \) and \( y = 0 \). Then \( x + y = 166 + 0 = 166 > 165 \).

This argument is not a correct proof of the statement. Explain why not.

The statement is about an arbitrary element \( y \) from \( \mathbb{N} \), and the proof works only for \( y = 0 \). The proof needs to work for an arbitrary element \( y \in \mathbb{N} \).

(e) [1 mark] Compute the numerical value of \( \sum_{i=-1}^{1} (2i - 1) \).

\[
\begin{align*}
&= (2(-1) - 1) + (2(0) - 1) + (2(1) - 1) \\
&= -3 - 1 + 1 \\
&= -3
\end{align*}
\]
2. [5 marks] Propositional logic.

(a) [3 marks] Write the truth table for the following formula. No rough work is required.

\[(p \Rightarrow q) \Rightarrow (r \Rightarrow p)\]

<table>
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<tr>
<th>P</th>
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<th>[(p \Rightarrow q) \Rightarrow (r \Rightarrow p)]</th>
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(b) [2 marks] Write a formula that is logically equivalent to the formula from Part (a) that only uses the operators: \(\neg\), \(\land\), \(\lor\). Show your work. You do not need to give the names of any equivalence rules. You do not need to simplify your final answer.

Reminder: Review and follow the guidelines stated on Page 1 of this test.

\[(p \Rightarrow q) \Rightarrow (r \Rightarrow p)\]

\[\iff\]
\[\neg (p \Rightarrow q) \lor (r \Rightarrow p)\]

\[\iff\]
\[(p \land \neg q) \lor (\neg r \lor p)\]

\[\uparrow \text{ not necessary} \quad \uparrow \text{ not necessary}\]
3. [6 marks] Translations. Consider the domain \( A = \{\text{all cats and dogs}\} \), and the predicate symbols \( C(x) \): “\( x \) is a cat”, \( D(x) \): “\( x \) is a dog”, \( T(x) \): “\( x \) has a tail”, \( B(x, y) \): “\( x \) bites \( y \)”, and \( L(x, y) \): “\( x \) likes \( y \)”

Using only these symbols and appropriate connectives and quantifiers, translate each sentence below. That is, give a natural English sentence that corresponds to each given symbolic sentence, and give a clear symbolic sentence that corresponds to each given English sentence. Quantifiers may only be used on the domain \( A \).

(a) [1 mark] Some cat does not have a tail.

\[
\exists x \in A, \; C(x) \land \neg T(x)
\]

(b) [1 mark] \( \forall x \in A, \; D(x) \Rightarrow T(x) \)

All dogs have tails.

(c) [1 mark] Some dog likes some cat.

\[
\exists x, y \in A, \; D(x) \land C(y) \land L(x, y)
\]

(d) [1 mark] Cats do not like dogs.

\[
\forall x, y \in A, \; (D(x) \land C(y)) \Rightarrow \neg L(y, x)
\]

(e) [2 marks] All cats with tails bite some dog.

\[
\forall x \in A, \; (C(x) \land T(x)) \Rightarrow (\exists y \in A, \; D(y) \land B(x, y))
\]

Consider the predicate $M(n): \forall k \in \mathbb{N}, n = 7k + 4$, where $n \in \mathbb{N}$.

(a) [1 mark] Give one example of a natural number $n$ for which $M(n)$ is True, and then show that $M(3n + 6)$ is also True for your choice of $n$. **HINT:** Try using $k = 0$ to determine your $n$.

Let $n = 7(0) + 4$ then $3n+6 = 3(4) + 6 = 18 = 7(2) + 4 = 7(\overline{2}) + 4$ for $k' = 2$.

(b) [4 marks] Prove or disprove the statement: $\forall n \in \mathbb{N}, M(n) \Rightarrow M(3n + 6)$.

If you choose to disprove the statement, you must start by writing its negation.

Discussion.

Proof.

Let $n \in \mathbb{N}$ and assume $M(n)$. That is, we assume that $\forall k \in \mathbb{N}, n = 7k + 4$. Let $k'$ be such that $n = 7k + 4$.

Consider $3n + 6 = 3(7k + 4) + 6 = 21k + 12 + 6 = 21k + 14 + 4 = 7(3k + 2) + 4$.

Let $k'' = 3k + 2$. Then $3n + 6 = 7k'' + 4$ and $M(3n + 6)$ is True, as required.
5. [9 marks] Working with sets and operations. Consider the set \( T = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\} \). Each element of \( T \) is a 2-tuple of real numbers. For example, \((1.0, 6.5) \in T\).

We can define the addition (+) and multiplication (*) operations on the set \( T \) as follows:
For \( x_1, x_2, y_1, y_2 \in \mathbb{R} \), we define

- \((x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)\)
- \((x_1, y_1) \times (x_2, y_2) = (x_1 \times x_2, y_1 \times y_2)\)

For example, \((1.4) + (2.5) = (3.9)\) and \((1.4) \times (2.5) = (2.20)\).

A set \( S \) is said to be closed under a given operation if performing that operation using elements of \( S \) always gives a result that is an element of \( S \). For example, \( \mathbb{N} \) is closed under addition, because the sum of any two natural numbers is itself a natural number. However, \( \mathbb{N} \) is not closed under subtraction, because, although 5 and 9 are both natural numbers, \( 5 - 9 = -4 \) is not.

Now consider the set \( U = \{(x, y) \mid (x, y) \in T \text{ and } x < y\} \), together with the operations + and \( \times \), as defined above.

(a) [1 mark] Give one example of an element of \( U \).

\((0,1) \) (or \((1,2)\), \((1,1.1)\), \((2, \pi)\))

(b) [1 mark] Translate the statement “The set \( U \) is closed under the operation +.” into predicate logic. Quantify over the set \( \mathbb{R} \).

\[ \forall x_1, x_2, y_1, y_2 \in \mathbb{R}, \quad x_1 < y_1 \land x_2 < y_2 \implies x_1 + x_2 < y_1 + y_2 \]

(c) [3 marks] Prove or disprove the statement: “The set \( U \) is closed under the operation +.”

If you choose to disprove the statement, you must start by writing its negation.

**Discussion.**

\[ \neg (b) \]

\[ \forall x_1, x_2, y_1, y_2 \in \mathbb{R}, \quad (x_1, y_1) \in U \land (x_2, y_2) \in U \implies (x_1 + x_2, y_1 + y_2) \notin U \]

**Proof.**

Let \( x_1, x_2, y_1, y_2 \in \mathbb{R} \) and assume \( x_1 < y_1 \) and \( x_2 < y_2 \).

Then \( x_1 + x_2 < y_1 + x_2 \) (since \( x_1 < y_1 \))

\[ < y_1 + y_2 \]

as required.

\[ \square \]
(d) [1 mark] Translate the statement “The set U is closed under the operation *.” into predicate logic.
Quantify over the set R.
\[ \forall x_1, x_2, y_1, y_2 \in \mathbb{R}, \; x_1 < y_1 \land x_2 < y_2 \Rightarrow x_1 x_2 < y_1 y_2 \]
alt: \[ \forall x_1, x_2, y_1, y_2 \in \mathbb{R}, \; (x_1, y_1) \in U \land (x_2, y_2) \in U \Rightarrow (x_1 x_2, y_1 y_2) \in U. \]

(e) [3 marks] Prove or disprove the following statement: “The set U is closed under the operation *.”
If you choose to disprove the statement, you must start by writing its negation.

**Discussion.**

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**Proof.** The statement is not True. The negation is
\[ \exists x_1, x_2, y_1, y_2 \in \mathbb{R}, \; x_1 < y_1 \land x_2 < y_2 \land x_1 x_2 \geq y_1 y_2 \]

Let \( x_1 = -1, \; x_2 = -1, \; y_1 = 0 \) and \( y_2 = 0. \)

Then \( x_1 < y_1 \) and \( x_2 < y_2 \).

Also, \( x_1 x_2 = (-1)(-1) \)
\[ = 1 \]
\[ > 0 \]
\[ = y_1 y_2 \), as neg'd. \]