• This examination has 4 questions. There are a total of 8 pages, DOUBLE-SIDED.

• Final expressions in predicate logic must have negation symbols (¬) applied only to predicates or propositional variables, e.g., ¬p or ¬Prime(x).

• You may not define your own propositional operators, predicates, or sets unless asked for in the question.

• In your proofs, you may always use definitions we have covered in this course. However, you may not use any external facts about these definitions unless they are given in the question.

• You may not use proofs by induction or contradiction on this midterm.

Take a deep breath.
This is your chance to show us how much you’ve learned.
We WANT to give you the credit that you’ve earned.
A number does not define you.
Good luck!

<table>
<thead>
<tr>
<th>Question</th>
<th>Grade</th>
<th>Out of</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>Q2</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>Q3</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>Q4</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td><strong>25</strong></td>
</tr>
</tbody>
</table>
1. [7 marks] Short answer questions. No justification is required for any parts of this question.

(a) [1 mark] Let \( S = \{5, 8, 10, 20\} \). Find a set \( S_1 \subseteq \mathbb{N} \) such that the following properties all hold:

\[ \forall x \in S_1, \forall y \in S, x < y \quad \text{and} \quad |S_1| = |S| \]

Solution

\( S_1 \) is any four-element subset of \( \{0, 1, 2, 3, 4\} \), e.g., \( S_1 = \{1, 2, 3, 4\} \).

(b) [2 marks] Let \( T = \{4, 6, 8, 10\} \). Find sets \( T_1 \subseteq T \) and \( T_2 \subseteq T \) such that the following properties all hold:

\[ |T_1 \cap T_2| = 0 \quad \text{and} \quad T_1 \cup T_2 = T \quad \text{and} \quad 2 \cdot |T_1 \times T_1| = |T_2 \times T| \]

Solution

\( T_1 \) is any two numbers from \( T \) and \( T_2 \) consists of the other two numbers, e.g., \( T_1 = \{4, 6\} \) and \( T_2 = \{8, 10\} \).

(c) [2 marks] Write down the truth table for the following expression in propositional logic. Intermediate columns of the truth table are not required, but can be included if you want.

\[(\neg p \lor q) \iff r\]

Solution

\[
\begin{array}{ccc|c}
 p & q & r & (\neg p \lor q) \iff r \\
 False & False & False & False \\
 False & False & True & True \\
 False & True & False & False \\
 False & True & True & True \\
 True & False & False & False \\
 True & False & True & False \\
 True & True & False & False \\
 True & True & True & True \\
\end{array}
\]

(d) [2 marks] Suppose we want to prove the following statement:

\[ \forall x \in \mathbb{N}, (\exists y \in \mathbb{N}, P(x, y)) \Rightarrow (\exists z \in \mathbb{N}, Q(x, z)) \]

Write the complete proof header for a proof, introducing all variables and assumptions. You may write statements like “Let \( d = \ldots \)” without filling in the blank. The last statement of your proof header should be “We will prove that...” where you clearly state what’s left to prove.
Solution

Let $x \in \mathbb{N}$.
Assume that there exists a $y$ such that $P(x, y)$ is true. Let $z = \underline{\quad}$.
We will prove that $Q(x, z)$ is true.
2. **[8 marks] Translations.** Let $P$ be the set of all people, and suppose we define the following predicates:

- $Teacher(x)$: “$x$ is a teacher”, where $x \in P$
- $Student(x)$: “$x$ is a student”, where $x \in P$
- $Remembers(x, y)$: “$x$ remembers $y$”, where $x, y \in P$

$(Remembers(x, y)$ does not mean the same thing as $Remembers(y, x)$)

Translate each of the following statements into predicate logic. No explanation is necessary. Do not define any of your own predicates or sets. You may use the $=$ and $\neq$ symbols to compare whether two people are the same.

(a) **[2 marks]** At least one student remembers at least one teacher.

**Solution**

$$\exists p_1, p_2 \in P, \ Student(p_1) \land Teacher(p_2) \land Remembers(p_1, p_2)$$

(b) **[2 marks]** Every teacher remembers himself/herself and only remembers himself/herself.

**Solution**

$$\forall t \in P, \ Teacher(t) \Rightarrow (\forall p \in P, \ Remembers(t, p) \iff t = p)$$

Or, $\forall t \in T, \ Teacher(t) \Rightarrow Remembers(t, t) \land (\forall p \in P, \ p \neq t \Rightarrow \neg Remembers(t, p))$

(c) **[2 marks]** If there is at least one teacher, then no one is a student.

**Solution**

$$(\exists p_1 \in P, \ Teacher(p_1)) \Rightarrow (\forall p_2 \in P, \ \neg Student(p_2))$$

(d) **[2 marks]** There is exactly one person who remembers every student.

**Solution**

$$\exists p_1 \in P, \ (\forall s \in P, \ Student(s) \Rightarrow Remembers(p_1, s)) \land \ (\forall p_2 \in P, \ (\forall s \in P, \ Student(s) \Rightarrow Remembers(p_2, s)) \Rightarrow p_2 = p_1)$$

Or, $\exists p_1 \in P, \ \forall p_2 \in P, \ (\forall s \in P, \ Student(s) \Rightarrow Remembers(p_2, s)) \iff p_1 = p_2$

The following is tempting, but incorrect:

$$\exists p_1 \in P, \ \forall s \in P, \ Student(s) \Rightarrow (\exists p_2 \in P, \ Remembers(p_1, s) \land \ Remembers(p_2, s))$$(\forall p_2 \in P, \ Remembers(p_2, s) \Rightarrow p_2 = p_1)$$
By using the same $s$ in the last part, we're saying that $p_2$ must equal $p_1$ if $p_2$ remembers any one student $s$ (rather than all students), since the implication $\text{Remembers}(p_2, s) \Rightarrow p_2 = p_1$ must be true for all $p_1$ and $p_2$. 
3. [5 marks] A proof about numbers.

Let \( n \in \mathbb{Z}^+ \). A sum of \( n \) consecutive integers is a summation of the form \( \sum_{i=0}^{n-1} (x + i) \), for some \( x \in \mathbb{Z} \).

(a) [1 mark] Translate the following statement into predicate logic: “For every odd positive integer \( n \), every sum of \( n \) consecutive integers is divisible by \( n \).”

Do not use the predicates \( | \) or \( \text{Odd} \), but instead expand their definitions in your statement (“odd” means \( n = 2k - 1 \) for some \( k \in \mathbb{Z} \)). You may use summation notation in your translation.

Solution

\[ \forall n \in \mathbb{Z}^+, (\exists k_1 \in \mathbb{Z}, n = 2k_1 - 1) \Rightarrow \left( \forall x \in \mathbb{Z}, \exists k_2 \in \mathbb{Z}, \sum_{i=0}^{n-1} (x + i) = nk_2 \right) \]

(b) [4 marks] Prove the statement from part (a) in the space below (you may also continue onto the next page). You may find the following formula helpful:

\[ \forall n \in \mathbb{Z}^+, \forall x \in \mathbb{Z}, \sum_{i=0}^{n-1} (x + i) = nx + \frac{n(n - 1)}{2} \]

Solution

Proof. Let \( n \in \mathbb{Z}^+ \) and assume that there exists a \( k_1 \in \mathbb{Z} \) such that \( n = 2k_1 - 1 \). Let \( x \in \mathbb{Z} \). Let \( k_2 = x + k_1 - 1 \). We will prove that \( \sum_{i=0}^{n-1} (x + i) = nk_2 \).

We can do this using a calculation, starting with the left-hand side.

\[
\sum_{i=0}^{n-1} (x + i) = nx + \frac{n(n - 1)}{2} \quad \text{(from the given formula)}
\]

\[
= n \left( x + \frac{n - 1}{2} \right)
\]

\[
= n \left( x + \frac{2k_1 - 2}{2} \right) \quad \text{(by our assumption that \( n \) is odd)}
\]

\[
= n(x + k_1 - 1)
\]

\[
= nk_2 \quad \text{(by our choice of \( k_2 \))}
\]

Recall that the floor of a number \( x \in \mathbb{R} \), denoted \( \lfloor x \rfloor \), is defined as the greatest integer that is less than or equal to \( x \). Consider the following statement:

For every \( y \in \mathbb{R} \geq 0 \) that is greater than 20, there exists a \( x \in \mathbb{R} \geq 0 \) that satisfies \( x \lfloor x \rfloor = y \).

(a) [1 mark] Write the negation of this statement in predicate logic.

Solution

\[ \exists y \in \mathbb{R} \geq 0, \ y > 20 \land (\forall x \in \mathbb{R} \geq 0, \ x \lfloor x \rfloor \neq y) \]

(b) [4 marks] Disprove the original statement by proving its negation in the space below.

Hint: do a proof by cases. The function \( x \lfloor x \rfloor \) has “jumps”.

Solution

Proof. Let \( y = 25 \). We'll prove that \( y > 20 \) and that for all \( x \in \mathbb{R} \geq 0 \), \( x \lfloor x \rfloor \neq y \). First, \( y = 25 > 20 \). Now let \( x \in \mathbb{R} \geq 0 \). We'll prove that \( x \lfloor x \rfloor \neq 25 \). We'll divide our proof into two cases.

Case 1: assume \( x \geq 3 \).

In this case, \( \lfloor x \rfloor \geq 3 \). So then \( x \lfloor x \rfloor \geq x^3 \), and since we assumed \( x \geq 3 \), \( x^3 \geq 3^3 = 27 \). And so \( x \lfloor x \rfloor \geq 27 \), and therefore \( x \lfloor x \rfloor \neq 25 \).

Case 2: assume \( x < 3 \).

In this case, \( \lfloor x \rfloor \leq 2 \) (note that for any \( x \) between 2 and 3 exclusive, its floor is 2). So then \( x \lfloor x \rfloor \leq x^2 \), and since we assumed \( x < 3 \), we know \( x^2 < 3^2 = 9 \). And so \( x \lfloor x \rfloor < 9 \), and therefore \( x \lfloor x \rfloor \neq 25 \). \( \square \)