1. [5 marks] Propositional logic.

(a) [3 marks] Write the truth table for the following formula. No rough work is required.

\[ \neg p \Rightarrow (q \land r) \]

Solution

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>( q \land r )</th>
<th>( \neg p \Rightarrow (q \land r) )</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
</tbody>
</table>

(b) [2 marks] Write a formula that is logically equivalent to the formula from Part (a) that only uses the operators: \( \neg, \land, \lor \). Show your work. You do not need to give the names of any equivalence rules.

Reminder: Review and follow the guidelines stated on Page 1 of this test.

Solution

\[ \neg p \Rightarrow (q \land r) \]

\[ \iff \neg p \lor (q \land r) \]

\[ \iff p \lor (q \land r) \]
2. [5 marks] **Statements in logic.**

A natural number whose square can be written in the form \( p^2 + q^2 \) for some positive integers \( p \) and \( q \) is called a *Pythagorean number*. For example, the natural number 5 is a Pythagorean number since \( 25 = 9 + 16 \) tells us that \( 5^2 = 3^2 + 4^2 \).

(a) [2 marks] Define a predicate \( \text{Pythagorean}(n) \), where \( n \) has domain \( \mathbb{N} \) (the set of natural numbers), that expresses the English statement:

"\( n \) is a Pythagorean number".

Use the notation of predicate logic in your solution, not English words.

**Solution**

\[
\text{Pythagorean}(n) : \ \exists \ p, q \in \mathbb{Z}, \ p > 0 \land q > 0 \land n^2 = p^2 + q^2, \quad \text{where} \ n \in \mathbb{N}.
\]

(b) [3 marks] Express using the language of predicate logic the English statement:

"There are infinitely many Pythagorean numbers."

You may use the predicates <, \( \leq \), = and \( \text{Pythagorean} \), but may not use any other predicate or function symbols.

Reminder: Review and follow the guidelines stated on Page 1 of this test.

**Solution**

\[
\forall \ x \in \mathbb{N}, \exists \ y \in \mathbb{N}, \ x < y \land \text{Pythagorean}(y)
\]

Can also say: \( \forall \ x \in \mathbb{N}, \exists \ y \in \mathbb{N}, \ x \leq y \land \text{Pythagorean}(y) \)

Or: \( \forall \ x \in \mathbb{N}, \text{Pythagorean}(x) \Rightarrow (\exists \ y \in \mathbb{N}, x < y \land \text{Pythagorean}(y)) \)

Or: \( \forall \ x \in \mathbb{N}, \exists \ y \in \mathbb{N}, \text{Pythagorean}(x) \Rightarrow (x < y \land \text{Pythagorean}(y)) \)
3. [5 marks] Proofs (inequalities). Consider the following statement: “There exists a natural number $n_0$ such that for every natural number $n$ greater than $n_0$, $8n^2 \leq n^3 - 20n$.”

(a) [1 mark] Translate the above statement into predicate logic. (Do not define your own set.)

Solution

$$\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n > n_0 \Rightarrow 8n^2 \leq n^3 - 20n$$

(b) [4 marks] Prove the statement. In your proof, any chains of calculations must follow a top-down order; don’t start with the inequality you’re trying to prove!

Write any rough work or intuition in the Discussion box, and write your formal proof in the Proof box. Your rough work/intuition will only be graded if your proof is not completely correct.

**HINT:** Don’t spend time trying to find the smallest possible value for $n_0$. Pick a large number.

Solution

Proof. Let $n_0 = 100$. Also let $n$ be an arbitrary natural number that is greater than $n_0$. We need to show that $8n^2 \leq n^3 - 20n$, or equivalently, $n^3 - 20n \geq 8n^2$.

Since $n > n_0$ and $n_0 = 100$, $n^3 = n \cdot n^2 \geq 100n^2$. Then

$$n^3 - 20n \geq 100n^2 - 20n$$
$$\geq 100n^2 - 20n^2$$
$$= 80n^2$$
$$\geq 8n^2.$$  

(There are many other valid sequences of steps.)
4. [5 marks] Proofs and disproofs.

Let \( x, y, m \in \mathbb{N} \). We say that \( m \) is a **common multiple** of \( x \) and \( y \) if and only if \( x \) divides \( m \) and \( y \) divides \( m \). We can define the predicate \( \text{IsCM}(x, y, m) \): \( \text{“} x \mid m \land y \mid m, \text{”} \) where \( x, y, m \in \mathbb{N} \).

We say that \( z \) is the **least common multiple** of \( x \) and \( y \) if and only if it is the smallest common multiple of \( x \) and \( y \), and in this case write \( z = \text{lcm}(x, y) \). For example, the least common multiple of 4 and 6 is 12. We can write \( 12 = \text{lcm}(4, 6) \).

(a) **[2 marks]** Use the language of predicate logic to fill in the blank below and complete the definition of the \( \text{lcm}(x, y) \) function. You may use the \( \text{IsCM}(x, y, m) \) predicate that was defined above.

\[
\forall x, y, z \in \mathbb{N}, \left( z = \text{lcm}(x, y) \Leftrightarrow (\text{IsCM}(x, y, z) \land \text{__________________}) \right)
\]

**Solution**

\[
\forall x, y, z \in \mathbb{N}, \left( z = \text{lcm}(x, y) \Leftrightarrow (\text{IsCM}(x, y, z) \land \forall w \in \mathbb{N}, \text{IsCM}(x, y, w) \Rightarrow z \leq w) \right)
\]

(b) **[3 marks]** Prove or disprove the following statement: \( \forall x, y, z \in \mathbb{N}, \ z = \text{lcm}(x, y) \Rightarrow z + 1 = \text{lcm}(x, y + 1) \).
If you choose to disprove the statement, you must start by writing its negation.

**Solution**

This statement is not True. Its negation is:

\[
\neg \left( \forall x, y, z \in \mathbb{N}, \ z = \text{lcm}(x, y) \Rightarrow z + 1 = \text{lcm}(x, y + 1) \right)
\]

\[
\iff \left( \exists x, y, z \in \mathbb{N}, \left( z = \text{lcm}(x, y) \right) \land \left( z + 1 \neq \text{lcm}(x, y + 1) \right) \right)
\]

**Proof.** Let \( x = 2, y = 3 \) and \( z = 6 \). We note that \( z = \text{lcm}(x, y) \). Also note that \( z + 1 = 7 \) and \( \text{lcm}(2, 4) = 4 \), and so \( \text{lcm}(x, y + 1) = 4 \). Since \( 7 \neq 4 \), we have \( z + 1 \neq \text{lcm}(x, y + 1) \).

(There are many other valid choices for \( x \) and \( y \).) \( \square \)