1. [5 marks] Propositional logic.

(a) [3 marks] Write the truth table for the following formula. No rough work is required.

\[ \neg(p \lor q) \Rightarrow r \]

**Solution**

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>( \neg(p \lor q) )</th>
<th>( \neg(p \lor q) \Rightarrow r )</th>
</tr>
</thead>
<tbody>
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</tr>
</tbody>
</table>

(b) [2 marks] Write a formula that is logically equivalent to the formula from Part (a) that only uses the operators: \( \neg, \land, \lor \). Show your work. You do **not** need to give the names of any equivalence rules.

Reminder: Review and follow the guidelines stated on Page 1 of this test.

**Solution**

\[ \neg(p \lor q) \Rightarrow r \]

\[ \iff \neg(p \lor q) \lor r \]

\[ \iff \neg(p \land \neg q) \lor r \]

\[ \iff (p \land \neg q) \lor r \]
2. [5 marks] Statements in logic.

A natural number that can be written in the form \( m \cdot m \) for some natural number \( m \) is called a *perfect square*. For example, the natural number 4 is a perfect square since \( 4 = 2 \cdot 2 \).

(a) [2 marks] Define a predicate \( \text{PerfectSquare}(n) \), where \( n \) has domain \( \mathbb{N} \) (the set of natural numbers), that expresses the English statement:

“\( n \) is a perfect square”.

Use the notation of predicate logic in your solution, not English words.

**Solution**

\[
\text{PerfectSquare}(n): \exists m \in \mathbb{N}, \ m^2 = n, \quad \text{where } n \in \mathbb{N}.
\]

(b) [3 marks] Express using the language of predicate logic the English statement:

“There is a smallest perfect square.”

You may use the predicates \( <, \leq, = \) and \( \text{PerfectSquare} \), but may not use any other predicate or function symbols.

Reminder: Review and follow the guidelines stated on Page 1 of this test.

**Solution**

\[
\exists x \in \mathbb{N}, \ \text{PerfectSquare}(x) \land (\forall y \in \mathbb{N}, \ \text{PerfectSquare}(y) \Rightarrow x \leq y).
\]
3. [5 marks] Proofs (inequalities). Consider the following statement: “There exists a natural number $n_0$ such that for every natural number $n$ greater than $n_0$, $9n^2 \leq n^3 - 10n$.”

(a) [1 mark] Translate the above statement into predicate logic. (Do not define your own set.)

**Solution**

$$\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, \ n > n_0 \ \Rightarrow \ 9n^2 \leq n^3 - 10n$$

(b) [4 marks] Prove the statement. In your proof, any chains of calculations must follow a top-down order; don’t start with the inequality you’re trying to prove!

Write any rough work or intuition in the Discussion box, and write your formal proof in the Proof box. Your rough work/intuition will only be graded if your proof is not completely correct.

**HINT:** Don’t spend time trying to find the smallest possible value for $n_0$. Pick a large number.

**Solution**

**Proof.** Let $n_0 = 200$. Also let $n$ be an arbitrary natural number that is greater than $n_0$. We need to show that $9n^2 \leq n^3 - 10n$, or equivalently, $n^3 - 10n \geq 9n^2$.

Since $n > n_0$ and $n_0 = 200$, $n^3 = n \cdot n^2 \geq 200n^2$. Then

$$n^3 - 10n \geq 200n^2 - 10n$$
$$\geq 200n^2 - 10n^2$$
$$= 190n^2$$
$$\geq 9n^2.$$  

(There are many other valid sequences of steps.)

□
4. [5 marks] Proofs and Disproofs.

Recall that a perfect square is a natural number that can be written in the form \( m \cdot m \) for some natural number \( m \). Let \( \text{PreviousPerfectSquare}(x) \) be a function from \( \mathbb{N} \) to \( \mathbb{N} \) that returns the largest perfect square that is less than or equal to \( x \).

(a) [2 marks] Use the language of predicate logic to fill in the blank below and complete the definition of the \( \text{PreviousPerfectSquare}(x) \) function. You may use the \( \text{PerfectSquare}(n) \) predicate that was defined in Question 2.

\[
\forall x, y \in \mathbb{N}, \left( y = \text{PreviousPerfectSquare}(x) \right) \iff \left( y \leq x \land \text{PerfectSquare}(y) \land \right)\]

Solution

\[
\forall x, y \in \mathbb{N}, \left( y = \text{PreviousPerfectSquare}(x) \right) \iff \left( y \leq x \land \text{PerfectSquare}(y) \land \forall z \in \mathbb{N}, (\text{PerfectSquare}(z) \land z \leq x) \rightarrow z \leq y \right)
\]

(b) [3 marks] Prove or disprove the following statement:
\[
\forall x, y \in \mathbb{N}, (y = \text{PreviousPerfectSquare}(x) \Rightarrow y^2 = \text{PreviousPerfectSquare}(x^2)).
\]

If you choose to disprove the statement, you must start by writing its negation.

Solution

This statement is not True. Its negation is:

\[
\neg \left( \forall x, y \in \mathbb{N}, (y = \text{PreviousPerfectSquare}(x) \Rightarrow y^2 = \text{PreviousPerfectSquare}(x^2)) \right)
\]

\[
\iff \exists x, y \in \mathbb{N}, (y = \text{PreviousPerfectSquare}(x)) \land (y^2 \neq \text{PreviousPerfectSquare}(x^2))
\]

Proof. Let \( x = 3 \) and \( y = 1 \). We note that \( y = \text{PreviousPerfectSquare}(x) \). Also \( y^2 = 1 \) and \( x^2 = 9 \), and so \( \text{PreviousPerfectSquare}(x^2) = 9 \). Since \( 1 \neq 9 \), we have \( y^2 \neq \text{PreviousPerfectSquare}(x^2) \).

(There are many other valid choices for \( x \) and \( y \).)