Name:

Student Number:

Please read the following guidelines carefully.

- Please print your name and student number on the front of the exam.
- This examination has 4 questions. There are a total of 6 pages, DOUBLE-SIDED.
- Answer questions clearly and completely. Provide justification unless explicitly asked not to.
- Unless stated otherwise, your formulas can use only the propositional connectives and quantifiers we have seen in class, arithmetic operators (like +, ×, and exponentiation), comparison operators (like = and >), and the divisibility and Prime predicates. You may not define your own sets or predicates unless asked to do so.
- All formulas must have negations applied directly to propositional variables or predicates. (e.g., ¬Prime(n)).
- You may not use induction for your proofs on this midterm.

Take a deep breath.

This is your chance to show us

How much you’ve learned.

We WANT to give you the credit

That you’ve earned.

A number does not define you.

<table>
<thead>
<tr>
<th>Question</th>
<th>Grade</th>
<th>Out of</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Q2</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Q3</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Q4</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>20</strong></td>
<td></td>
</tr>
</tbody>
</table>
1. [5 marks] Propositional logic.

(a) [3 marks] Write the truth table for the following formula. No rough work is required.

\((\neg p \lor q) \Rightarrow r\)

(b) [2 marks] Write a formula that is logically equivalent to the formula from Part (a) that only uses the operators: \(\neg, \land, \lor\). Show your work. You do not need to give the names of any equivalence rules.

Reminder: Review and follow the guidelines stated on Page 1 of this test.
2. [5 marks] Statements in logic.

A natural number that can be written in the form \( m \cdot m \) for some natural number \( m \) is called a perfect square. For example, the natural number 4 is a perfect square since \( 4 = 2 \cdot 2 \).

(a) [2 marks] Define a predicate \( \text{PerfectSquare}(n) \), where \( n \) has domain \( \mathbb{N} \) (the set of natural numbers), that expresses the English statement:

“\( n \) is a perfect square”.

Use the notation of predicate logic in your solution, not English words.

(b) [3 marks] Express using the language of predicate logic the English statement:

“There is a smallest perfect square.”

You may use the predicates \(<, \leq, =\) and \( \text{PerfectSquare} \), but may not use any other predicate or function symbols.

Reminder: Review and follow the guidelines stated on Page 1 of this test.
3. [5 marks] Proofs (inequalities). Consider the following statement: “There exists a natural number $n_0$ such that for every natural number $n$ greater than $n_0$, $9n^2 \leq n^3 - 10n$.”

(a) [1 mark] Translate the above statement into predicate logic. (Do not define your own set.)

(b) [4 marks] Prove the statement. In your proof, any chains of calculations must follow a top-down order; don’t start with the inequality you’re trying to prove!
Write any rough work or intuition in the Discussion box, and write your formal proof in the Proof box. Your rough work/intuition will only be graded if your proof is not completely correct.
**HINT:** Don’t spend time trying to find the smallest possible value for $n_0$. Pick a large number.

**Discussion.**

**Proof.**
4. **[5 marks] Proofs and Disproofs.**

Recall that a *perfect square* is a natural number that can be written in the form $m \cdot m$ for some natural number $m$. Let $PreviousPerfectSquare(x)$ be a function from $\mathbb{N}$ to $\mathbb{N}$ that returns the largest perfect square that is less than or equal to $x$.

(a) **[2 marks]** Use the language of predicate logic to fill in the blank below and complete the definition of the $PreviousPerfectSquare(x)$ function. You may use the $PerfectSquare(n)$ predicate that was defined in Question 2.

$$\forall x, y \in \mathbb{N}, (y = PreviousPerfectSquare(x) \iff (y \leq x \land PerfectSquare(y) \land \text{[ ]}))$$

(b) **[3 marks]** Prove or disprove the following statement:

$$\forall x, y \in \mathbb{N}, (y = PreviousPerfectSquare(x) \Rightarrow y^2 = PreviousPerfectSquare(x^2)).$$

If you choose to disprove the statement, you must start by writing its negation.
Midterm 1, Version 2, CSC165H1S

Use this page for rough work. If you want work on this page to be marked, please indicate this clearly at the location of the original question.