Please read the following guidelines carefully.

- Please print your name and student number on the front of the exam.
- This examination has 4 questions. There are a total of 6 pages, DOUBLE-SIDED.
- Answer questions clearly and completely. Provide justification unless explicitly asked not to.
- Unless stated otherwise, your formulas can use only the propositional connectives and quantifiers we have seen in class, arithmetic operators (like +, ×, and exponentiation), comparison operators (like = and >), and the divisibility and Prime predicates. You may not define your own sets or predicates unless asked to do so.
- All formulas must have negations applied directly to propositional variables or predicates. (e.g., ¬Prime(n)).
- You may not use induction for your proofs on this midterm.

Take a deep breath.
This is your chance to show us
How much you’ve learned.
We WANT to give you the credit
That you’ve earned.
A number does not define you.

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1. [5 marks] Propositional logic.

(a) [3 marks] Write the truth table for the following formula. No rough work is required.

\[(p \land \neg q) \Rightarrow r\]

(b) [2 marks] Write a formula that is logically equivalent to the formula from Part (a) that only uses the operators: \(\neg, \land, \lor\). Show your work. You do not need to give the names of any equivalence rules.

Reminder: Review and follow the guidelines stated on Page 1 of this test.
2. [5 marks] Statements in logic.

A real number that can be written in the form \( \frac{p}{q} \) for some integers \( p \) and \( q \) is called a rational number. For example, the real number 0.2 is a rational number since \( 0.2 = \frac{1}{5} \).

(a) [2 marks] Define a predicate \( \text{Rational}(x) \), where \( x \) has domain \( \mathbb{R} \) (the set of real numbers), that expresses the English statement:

“\( x \) is a rational number”.

Use the notation of predicate logic in your solution, not English words. Do not use the set \( \mathbb{Q} \) in your solution.

(b) [3 marks] Express using the language of predicate logic the English statement:

“There is a largest rational number.”

You may use the predicates \(<\), \(\leq\), \(=\) and \(\text{Rational} \), but may not use any other predicate or function symbols. You may not use the set \( \mathbb{Q} \) in your solution.

Reminder: Review and follow the guidelines stated on Page 1 of this test.
3. [5 marks] Proofs (inequalities). Consider the following statement: “There exists a natural number $n_0$ such that for every natural number $n$ greater than $n_0$, $\frac{n^3}{10} \geq 7n^2 + 13$.”

(a) [1 mark] Translate the above statement into predicate logic. (Do not define your own set.)

(b) [4 marks] Prove the statement. In your proof, any chains of calculations must follow a top-down order; don’t start with the inequality you’re trying to prove!
Write any rough work or intuition in the Discussion box, and write your formal proof in the Proof box. Your rough work/intuition will only be graded if your proof is not completely correct.
HINT: Don’t spend time trying to find the smallest possible value for $n_0$. Pick a large number.

Discussion.

Proof.
4. [5 marks] **Proofs and disproofs.**

Recall that the floor of a real number $x$, denoted $\lfloor x \rfloor$, is defined as the largest integer that is less than or equal to $x$.

(a) [2 marks] Use the language of predicate logic to fill in the blank below and complete the definition of the $\lfloor x \rfloor$ function:

$$\forall x, y \in \mathbb{R}, \left( y = \lfloor x \rfloor \iff (y \in \mathbb{Z} \land y \leq x \land \text{____________________________}) \right)$$

(b) [3 marks] Prove or disprove the following statement: $\forall x, y \in \mathbb{R}, \left( y > 0 \Rightarrow \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor \right)$.

If you choose to disprove the statement, you must start by writing its negation.

**Discussion.**

**Proof.**
Use this page for rough work. If you want work on this page to be marked, please indicate this clearly at the location of the original question.