Learning Objectives

By the end of this worksheet, you will:

- Write proofs using simple induction with different starting base cases.
- Write proofs using simple induction within the scope of a larger proof.

1. **Induction (summations).** If marbles are arranged to form an equilateral triangle shape, with \( n \) marbles on each side, a total of \( \sum_{i=1}^{n} i \) marbles will be required.

In lecture, we proved that \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \). For each \( n \in \mathbb{N} \), let \( T_n = \frac{n(n+1)}{2} \); these numbers are usually called the **triangular numbers**. Use induction to prove that

\[
\forall n \in \mathbb{N}, \sum_{j=0}^{n} T_j = \frac{n(n+1)(n+2)}{6}
\]

**Solution**

Let us start by defining the predicate

\[
P(n) : \sum_{j=0}^{n} T_j = \frac{n(n+1)(n+2)}{6}
\]

We need to prove that \( \forall n \in \mathbb{N}, P(n) \).

**Proof. Base case:** let \( n = 0 \). We want to prove \( P(0) \). Then we can calculate:

\[
\sum_{j=0}^{n} T_j = \sum_{j=0}^{0} T_j = T_0 = \frac{0(0+1)}{2} = 0
\]
And also $\frac{0(0+1)(0+2)}{6} = 0$.

**Induction step:** Let $k \in \mathbb{N}$ and assume $P(k)$, i.e., that $\sum_{j=0}^{k} T_j = \frac{k(k+1)(k+2)}{6}$. We want to prove $P(k+1)$, i.e., that $\sum_{j=0}^{k+1} T_j = \frac{(k+1)(k+2)(k+3)}{6}$.

We’ll calculate starting from the left side and show that it equals the right side.

$$\sum_{j=0}^{k+1} T_j = \left( \sum_{j=0}^{k} T_j \right) + T_{k+1}$$

$$= \frac{k(k+1)(k+2)}{6} + T_{k+1} \quad \text{(by our assumption of } P(k)\text{)}$$

$$= \frac{k(k+1)(k+2) + (k+1)(k+2)}{6} \quad \text{(by the definition of } T_{k+1}\text{)}$$

$$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{6}$$

$$= \frac{(k+1)(k+2)(k+3)}{6}$$

\[ \square \]
2. **Induction (inequalities).** Consider the statement:

For every positive real number \( x \) and every natural number \( n \), \((1 + x)^n \geq 1 + nx\).

We can express the statement using the notation of predicate logic as:

\[
\forall x \in \mathbb{R}^+, \forall n \in \mathbb{N}, (1 + x)^n \geq 1 + nx
\]

Notice that in this statement, there are two universally-quantified variables: \( n \) and \( x \). Prove the statement is true using the following approach:

(a) Use the standard proof structure to introduce \( x \).
(b) When proving the \((\forall n \in \mathbb{N}, (1 + x)^n \geq 1 + nx)\), do induction on \( n \).

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**Solution**

*Proof.* Let \( x \in \mathbb{R}^+ \). We’ll prove that for all \( n \in \mathbb{N}, (1 + x)^n \geq 1 + nx \) by induction.

**Base case:** Let \( n = 0 \).

Then \((1 + x)^0 = 1\) and \( 1 + nx = 1 \). So then \((1 + x)^0 \geq 1 + nx \).

**Induction step:** Let \( k \in \mathbb{N} \), and assume that \((1 + x)^k \geq 1 + kx \). We want to prove that \((1 + x)^{k+1} \geq 1 + (k+1)x \).

We’ll start with the quantity on the left, and show that it’s \( \geq \) the quantity on the right.

\[
(1 + x)^{k+1} = (1 + x)^k(1 + x) \\
\geq (1 + kx)(1 + x) \quad \text{(by our assumption)} \\
= 1 + kx + x + kx^2 \\
\geq 1 + kx + x \quad \text{(since } kx^2 \geq 0) \\
= 1 + (k + 1)x
\]

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1 Your predicate \( P(n) \) that you want to prove will also contain the variable \( x \) – that’s okay, since when we do the induction proof, \( x \) has already been defined.
3. **Changing the starting number.** Recall that you previously proved that $\forall n \in \mathbb{N}, \ n \leq 2^n$ using induction.

   (a) First, use trial an error to fill in the blank to make the following statement true – try finding the *smallest natural number* that works!

   $\forall n \in \mathbb{N}, n \geq \underline{\quad} \Rightarrow 30n \leq 2^n$

   **Solution**

   $\forall n \in \mathbb{N}, n \geq 8 \Rightarrow 30n \leq 2^n$.

   (b) Now, prove the completed statement using induction. Be careful about how you choose your base case!

   **Solution**

   \[
   \begin{align*}
   \text{Proof. Base case:} & \quad \text{Let } n = 8. \\
   \text{Then } 30n &= 240, \text{ and } 2^n = 256. \text{ So } 30n \leq 2^n. \\
   \text{Induction step:} & \quad \text{Let } k \in \mathbb{N}. \text{ Assume that } k \geq 8, \text{ and that } 30k \leq 2^k. \text{ We want to prove that } 30(k+1) \leq 2^{k+1}.
   \end{align*}
   \]

   Since $8 \leq k$, we know that $256 \leq 2^k$ (raising 2 to the power of either side). Our assumption tells us that $30k \leq 2^k$. Adding these two inequalities yields:

   \[
   \begin{align*}
   30k + 256 &\leq 2^k + 2^k \\
   30k + 256 &\leq 2^{k+1} \\
   30k + 30 &\leq 2^{k+1} \quad \text{(since } 30 \leq 256) \\
   30(k+1) &\leq 2^{k+1}
   \end{align*}
   \]