Learning Objectives

By the end of this worksheet, you will:

- Have been introduced to bipartite graphs.

1. Bipartite graphs. A bipartite graph is a graph $G = (V, E)$ that satisfies the following properties:
   
   (a) There exist subsets $V_1, V_2 \subset V$ such that $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, and $V_1$ and $V_2$ form a partition of $V$.
   
   (b) Every edge in $E$ has exactly one endpoint in $V_1$, and exactly one endpoint in $V_2$. (That is, no two vertices in $V_1$ are adjacent, and no two vertices in $V_2$ are adjacent.)

When $G$ is bipartite, we call the partitions $V_1$ and $V_2$ a bipartition of $G$.

(a) Prove that the following graph $G = (V, E)$ is bipartite.

   $$V = \{1, 2, 3, 4, 5, 6\} \quad \text{and} \quad E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\}$$

**Solution**

Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$. Then $V_1$ and $V_2$ together provide a partition of $V$, as $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and neither $V_1$ nor $V_2$ is empty.

Note that all of the vertex labels in $V_1$ are odd numbers and all of the vertex labels in $V_2$ are even numbers.

Each of the edges $(1, 2)$, $(1, 6)$, $(2, 3)$, $(3, 4)$, $(4, 5)$, and $(5, 6)$, has one endpoint that with a vertex label that is an odd number and one that is an even number.

(b) Let $m$ and $n$ be positive integers. A complete bipartite graph on $(m, n)$ vertices is a graph $G = (V, E)$ that satisfies the following properties:

   i. There exist subsets $V_1, V_2 \subset V$ such that $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, and $V_1$ and $V_2$ form a partition of $V$.
   
   ii. Every edge in $E$ has exactly one endpoint in $V_1$, and exactly one endpoint in $V_2$. (That is, no two vertices in $V_1$ are adjacent, and no two vertices in $V_2$ are adjacent.)
   
   iii. (new) $|V_1| = m$ and $|V_2| = n$.
   
   iv. (new) For all vertices $u \in V_1$ and $w \in V_2$, $u$ and $w$ are adjacent.

How many edges are in a complete bipartite graph on $(m, n)$ vertices? Your answer will depend on $m$ and $n$. Explain your answer.

**Solution**

Let $G = (V, E)$ be a complete bipartite graph on $(m, n)$ vertices, with bipartition $V_1, V_2$, and $|V_1| = m$ and $|V_2| = n$.

Then each vertex $u \in V_1$ appears as an endpoint in $n$ edges in $E$, since it has an edge to each of the $n$ vertices in $V_2$. As there are $m$ vertices in $V_1$ and the previous statement is true for each of them, we know that there are at least $mn$ edges in $E$.

But, since there are no edges between vertices in $V_1$ and no edges between vertices in $V_2$, there are no other edges to count.

And so we can conclude that the number of edges in a complete bipartite graph on $(m, n)$ vertices is $mn$.

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1That is, $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$. 