Learning Objectives

By the end of this worksheet, you will:

• Have been introduced to bipartite graphs.

1. Bipartite graphs. A bipartite graph is a graph \( G = (V, E) \) that satisfies the following properties:

   (a) There exist subsets \( V_1, V_2 \subseteq V \) such that \( V_1 \neq \emptyset \), \( V_2 \neq \emptyset \), and \( V_1 \) and \( V_2 \) form a partition of \( V \)\(^\dagger\).

   (b) Every edge in \( E \) has exactly one endpoint in \( V_1 \), and exactly one endpoint in \( V_2 \). (That is, no two vertices in \( V_1 \) are adjacent, and no two vertices in \( V_2 \) are adjacent.)

When \( G \) is bipartite, we call the partitions \( V_1 \) and \( V_2 \) a bipartition of \( G \).

(a) Prove that the following graph \( G = (V, E) \) is bipartite.

\[
V = \{1, 2, 3, 4, 5, 6\} \quad \text{and} \quad E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\}
\]

(b) Let \( m \) and \( n \) be positive integers. A complete bipartite graph on \((m, n)\) vertices is a graph \( G = (V, E) \) that satisfies the following properties:

   i. There exist subsets \( V_1, V_2 \subseteq V \) such that \( V_1 \neq \emptyset \), \( V_2 \neq \emptyset \), and \( V_1 \) and \( V_2 \) form a partition of \( V \).

   ii. Every edge in \( E \) has exactly one endpoint in \( V_1 \), and exactly one endpoint in \( V_2 \). (That is, no two vertices in \( V_1 \) are adjacent, and no two vertices in \( V_2 \) are adjacent.)

   iii. (new) \( |V_1| = m \) and \( |V_2| = n \).

   iv. (new) For all vertices \( u \in V_1 \) and \( w \in V_2 \), \( u \) and \( w \) are adjacent.

How many edges are in a complete bipartite graph on \((m, n)\) vertices? Your answer will depend on \( m \) and \( n \). Explain your answer.

\(^\dagger\text{That is, } V_1 \cup V_2 = V \text{ and } V_1 \cap V_2 = \emptyset.\)