$G = (V, E)$

Write a description for graph $G$:

$$V = \{A, B, C, D, E\}$$

$$E = \\{ (A, B), (A, C), (C, D), (A, D) \}$$

- The number of vertices in $G$
  $$|V| = 5$$
- The number of edges in $G$
  $$|E| = 4$$

Can prove Thm about graphs.
e.g. Let $G = (V, E)$ be an arbitrary graph. Then $|E| \leq \frac{|V|(|V|-1)}{2}$

Proof. $orall G = (V, E) \in G$, $|E| \leq \frac{|V|(|V|-1)}{2}$

$G = \text{the set of all possible graphs}$

Let $G = (V, E)$ be an arbitrary graph.

Each edge in $G$ consists of pairs of distinct vertices from $V$, where order does not matter. The maximum number of edges is the same as the number of subsets of $V$ of size 2, which is $\frac{|V|(|V|-1)}{2}$. And so

$|E| \leq \frac{|V|(|V|-1)}{2}$.

Beware of double counting $(v_i, v_j) = (v_j, v_i)$

Other concepts:
- Can we get to vertex $v_j$?
How many vertices do we go through in trip from \( v_i \) to \( v_j \)?

(Could be multiple routes)

- Can we get from \( v_i \) to \( v_j \) with one edge?
- Can we get to all vertices \( v_j \) starting from \( v_i \)?

**Terminology**

Let \( G = (V, E) \).

**Def.**

Let \( v_1, v_2 \in V \). We say that \( v_1 \) and \( v_2 \) are adjacent if and only if

\[(v_1, v_2) \in E.\]

(also say \( v_1, v_2 \) are neighbours)

**Def.** Let \( u, u' \in V \)

A path between \( u \) and \( u' \) is a sequence of distinct vertices

\[v_0, v_1, v_2, \ldots, v_k \in V\]

that satisfy the properties

\[v_0 = u \quad \text{and} \quad v_k = u' \quad (\text{end points of path})\]

- each consecutive pair of vertices are
adjacent

\((u_i, u_{i+1}) \in E\)

for \(i = 0, 1, 2, ..., k-1\).

u

\[ \begin{array}{c}
  u_0 \quad u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_k \quad u_{k-1} \\
\end{array} \]

"distinct" says

u

\[ \begin{array}{c}
  u_0 \quad u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_k \quad u_{k-1} \\
\end{array} \]

- valid path

- not a valid path

- the length of a path: the \# of edges
  used by the sequence

- can a path have length 0?
  yes. allow \( u = u' \)

- there can be more than one path
  between \( u \) and \( u' \)

- there can be paths of different
  lengths from \( u \) to \( u' \)

- possible that there is no path
  from \( u \) to \( u' \)

- the distance from \( u \) to \( u' \)
  is the length of the shortest
path from \( u \) to \( u' \)

- if no path, distance is \( \infty \)

We say that vertices \( u \) and \( u' \) are connected if and only if \( \exists \) a path from \( u \) to \( u' \).

The graph \( G = (V, E) \) is connected if and only if for all pairs of vertices \( u, v \in V \), \( u \) and \( v \) are connected.

**Predicate:**

\[ \text{conn} \left( G = (V, E), u, v \right) : \]

" \( u \) and \( v \) are connected vertices in \( G \)". Where \( G = (V, E) \), \( u, v \in V \).

**Facts:**

1. \[ \text{conn}(G, u, v) \Rightarrow \text{conn}(G, u, u) \]

Prove \( \exists \) path from \( u \) to \( u \)

Path is: \( u, v, u, v, \ldots, u, v, u, v \)

Prove \( \exists \) path from \( u \) to \( u \)
Transitivity

\[ \text{conn}(G, u, v) \land \text{conn}(G, v, w) \Rightarrow \text{conn}(G, u, w) \]

Proof.

Assume there is a path

\[ a_0, a_1, a_2, \ldots, a_m, a_m, a_n, \ldots, a_n, u, v, w \]

\[ b_0, b_1, b_2, \ldots, b_{n-1}, b_n, \ldots, b_n, u, \ldots, a_n, v, w \]

Want to prove there is a path in G from u to w.

Case: a's and b's are distinct.

Then path is

\[ a_0, a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n \]

Case: a's and b's are not distinct:

\[ a_0, a_i = b_k, \ldots, a_n, a_i = b_k, \ldots, b_n, \ldots, a_n, v, w \]

Pick smallest i such that \( a_i = b_k \)
The next questions:

- is there a \( M_1 \) s.t. if \( |E| > M_1 \),
  
  \( G \) must be connected?

- is there a \( M_2 \) s.t. \( |E| < M_2 \),
  
  \( G \) must not be connected?

\[ G \text{ is not connected} \quad \text{if } |E| < M_2 \]
\[ G \text{ may or may not be connected} \quad \text{if } |E| = M_2 \]
\[ G \text{ is connected} \quad \text{if } |E| > M_2 \]

- it turns out that

\[ M_1 = \frac{(1v-1)(1v-2)}{2} \]

[see course notes]

\[ M_2 = ? \]

next time!