Reminder of Induction Principle

- to prove a statement of the form

\[ \forall n \in \mathbb{N}, \ P(n) \]

Variant: \[ \forall n \in \mathbb{N}, \ n \geq m \implies P(n) \] for some

\[ m \in \mathbb{N} \]

Idea:

If we can prove a base case \( P(0) \)

and can also prove \( P(0) \Rightarrow P(1) \) \[ \therefore P(1) \]

\[ P(1) \Rightarrow P(2) \] \[ \therefore P(2) \]

... can prove \( \forall k \in \mathbb{N}, \ P(k) \Rightarrow P(k+1) \)
\((P(0) \land (\forall k \in \mathbb{N}, P(k) \implies P(k+1))) \implies (\forall n \in \mathbb{N}, n > m \implies P(n))\)

**Variant:**
\[P(m) \land (\forall k \in \mathbb{N}, (k > m \land P(k)) \implies P(k+1))\]
\[\implies (\forall n \in \mathbb{N}, n > m \implies P(n))\]

\((\exists k \in \mathbb{N})\]

**Example:** The sum of the first \(n\) odd natural numbers is a perfect square.
\[L \geq k^2\] for some \(k\).

**Translate:**
\[
A \quad n \in \mathbb{N}, \quad \sum_{i=1}^{n} (2i-1) = k^2
\]

<table>
<thead>
<tr>
<th>Discuss</th>
<th>Sum of first (n) odd natural numbers</th>
<th>(n^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
</tr>
<tr>
<td>(1)</td>
<td>(1)</td>
<td>(1)</td>
</tr>
<tr>
<td>(2)</td>
<td>(1+3 = 4)</td>
<td>(4)</td>
</tr>
<tr>
<td>(3)</td>
<td>(1+3+5 = 9)</td>
<td>(9)</td>
</tr>
<tr>
<td>(4)</td>
<td>(1+3+5+7 = 16)</td>
<td>(16)</td>
</tr>
</tbody>
</table>

**Suggests:** take \(k = n\)
Prove $\forall n \in \mathbb{N}$, $\sum_{i=1}^{n} (2i-1) = n^2$

**Proof:** Let $P(n)$: $\sum_{i=1}^{n} (2i-1) = n^2$, where $n \in \mathbb{N}$.

**Base Case:** Prove $P(0)$.

\[
\sum_{i=1}^{0} (2i-1) = \sum_{i=1}^{0} (2i-1) = 0 = (0)^2 = n^2
\]

So $P(0)$ is True.

**Inductive Step:** We need to prove $\forall k \in \mathbb{N}$, $P(k) \implies P(k+1)$.

**Proof:** Let $k \in \mathbb{N}$ and assume $P(k)$.

Then we know $\sum_{i=1}^{k} (2i-1) = k^2$.

We need to prove $P(k+1)$.
\[
\sum_{i=1}^{k+1} (2i-1) = (k+1)^2
\]

Now \[
\sum_{i=1}^{k} (2i-1) = 1 + 3 + 5 \ldots + (2k-1) = (k^2)
\]

by induction hypothesis \[
\Rightarrow \sum_{i=1}^{k+1} (2i-1) + 2(k+1) = k^2 + 2k + 1
\]

goal \[
= (k+1)^2
\]

and so \( P(k+1) \).

\[
\therefore \forall k \in \mathbb{N}, \ P(k) \Rightarrow P(k+1)
\]

Hence, by the principle of simple induction

\[
\forall n \in \mathbb{N}, \ \sum_{i=1}^{n} (2i-1) = n^2
\]

Example Prove that \[
\forall x, y \in \mathbb{Z}, \ \forall n \in \mathbb{N}, \ 5 \mid (x-y) \Rightarrow 5 \mid (x^n - y^n)
\]

usual proof try an inductive proof.
Proof: Let \( x, y \in \mathbb{Z} \).

Prove \( \forall n \in \mathbb{N}, \, S \mid (x-y) = S \mid (x^n - y^n) \)

Define the predicate \( P(n) : \, S \mid (x-y) = S \mid (x^n - y^n) \),

where \( n \in \mathbb{N} \).

and the prove \( P(0) \land (\forall k \in \mathbb{N}, P(k) \implies P(k+1)) \)

base case: Prove \( P(0) \).

Let \( n = 0 \) and assume \( S \mid (x - y) \)

so that \( x - y = 5k \) for some \( k \in \mathbb{Z} \).

Need to prove \( S \mid (x^0 - y^0) \)

\( x^0 - y^0 = 1 - 1 = 0 \)

so \( P(0) \) is True.

inductive step: Let \( k \in \mathbb{N} \) and assume \( P(k) \)

That is \( S \mid (x-y) \implies S \mid (x^k - y^k) \)
Want to prove \( S \mid (x,y) \Rightarrow S \mid (x^{k+1} - y^{k+1}) \)

Assume \( S \mid (x,y) \)

Let \( c \in \mathbb{Z} \) be such that \((x,y) = Sc\)

Since we assumed \( S \mid (x,y) \), then by inductive hypothesis we know \( S \mid (x^k - y^k) \)

Let \( d \in \mathbb{Z} \) be such that \((x^k - y^k) = 5d\)

Now consider \( x^{k+1} - y^{k+1} \)

\[
= x \cdot x^k - y \cdot y^k
\]

Knows something about \((x,y)\) and \((x^k - y^k)\)

\[
= x(x^k - y^k) + xy^k - y^k
\]

\[
= x(x^k - y^k) + (x-y)y^k
\]

\[
= 5d + Scy^k
\]

\[
= S(xd + cy^k)
\]

Goal \( \Rightarrow \) \( S \cdot e \) for \( e = xd + cy^k \)

\( e \in \mathbb{Z} \).

and so \( P(k+1) \).

\( \square \)