Term Test 1
Thu Oct 11
115 - 230
Ex 100

to end Ch 2

\text{Prime}(p) : "p > 1 \land (\forall d \in \mathbb{N}, d \mid p = (d = 1 \lor d = p)),"

where \( p \in \mathbb{N} \)

defined\n
\text{Atomic}(n) : "\forall a, b \in \mathbb{N}, n \times a \land n \times b \Rightarrow n \times ab,"

where \( n \in \mathbb{N} \).

Explore:

(\ast) \ \forall n \in \mathbb{N}, \text{Prime}(n) \iff (n > 1 \land \text{Atomic}(n))

\( p \iff q \) means \( p \Rightarrow q \land q \Rightarrow p \)

We will prove (\ast) by proving:

\( \forall n \in \mathbb{N}, \text{Prime}(n) \Rightarrow (n > 1 \land \text{Atomic}(n)) \)
We started with (2) and considered its contrapositive:

\[ \forall n \in \mathbb{N}, \neg \text{Prime}(n) \Rightarrow (n \leq 1 \lor \neg \text{Atomic}(n)) \]

(small final details reconsidered in today's worksheet)

Now prove (1):

\[ \forall n \in \mathbb{N}, \text{Prime}(n) \Rightarrow (n > 1 \land \text{Atomic}(n)) \]

Proof will rely on these external facts:

Claim 1: \( \forall n, m \in \mathbb{N}, \text{Prime}(n) \land n \times m \Rightarrow (\exists r, s \in \mathbb{Z}, rn + sm = 1) \)

Claim 2: \( \forall n, m \in \mathbb{N}, \text{Prime}(n) \land (\exists r, s \in \mathbb{Z}, rn + sm = 1) \Rightarrow n \times m \)

Discuss:

- let \( n \) be an arbitrary prime number
- need to show \( n > 1 \) since \( n \) is prime
and

$\text{Atomic}(n)$

\[ \text{need to prove } \text{Atomic}(n) \]
\[ \forall a, b \in \mathbb{N}, \, n \mid a \land n \mid b \Rightarrow n \mid (ab) \]

Let $a, b \in \mathbb{N}$ and assume $n \mid a$ and $n \mid b$.

Need to show $n \mid (ab)$.

Know:

$\text{Prime}(n)$, $n \mid a$, $n \mid b$

apply claim 1:

\[ r \cdot n + s \cdot a = 1 \tag{*} \text{ for some } r, s \in \mathbb{Z}. \]

Similarly

\[ r_2 \cdot n + s_2 \cdot b = 1 \tag{*} \text{ for some } r_2, s_2 \in \mathbb{Z}. \]

Want to show:

$n \mid (ab)$

\[ \text{The conclusion in claim 2} \]
\[ \text{so try to show hypothesis True} \]

to apply claim 2, would need

\[ r_3 \cdot n + s_3 \cdot (ab) = 1 \text{ for some } r_3, s_3 \in \mathbb{Z}. \]

multiply \(*\) get

\[ (r \cdot n + s \cdot a) (r_2 \cdot n + s_2 \cdot b) = 1 \cdot 1 \]

\[ r_1 r_2 n^2 + s_1 s_2 ab + r_1 s_2 n + s_1 r_2 b = 1 \]

factor out $n$
\((^\downarrow \ \downarrow)^n + (\downarrow \ \downarrow) ab = 1\)
\[\text{call it } r_3 \quad s_3\]

Proof.

Let \(n \in \mathbb{N}\) and assume that \(n\) is prime.
We want to prove that \(n > 1\) and that \(\text{Atomic}(n)\) is True.

Since \(n\) is prime, we know \(n > 1\). To show that \(\text{Atomic}(n)\) is True, we need to prove that
\[\forall a, b \in \mathbb{N}, \ n \mid x_a \land n \mid x_b \implies n \mid x_{ab}\]

Let \(a, b \in \mathbb{N}\) and assume \(n \mid x_a\) and \(n \mid x_b\).

By claim 1, since \(n\) is prime, we know that there are \(r_1, s_1, r_2, s_2 \in \mathbb{Z}\) such that
\[r_1 n + s_1 a = 1\]
\[r_2 n + s_2 b = 1\]

Multiplying these two equations gives
\[(r_1 n + s_1 a)(r_2 n + s_2 b) = 1.1\]
\[ r_1 r_2 n^2 + s_1 a r_2 n + r_1 n s_2 b + s_1 s_2 a b = 1 \]

\[(r_1 r_2 n + s_1 a r_2 + r_1 s_2 b) n + (s_1 s_2) a b = 1\]

\[ r_3 n + s_3 (a b) = 1 \]

for \( r_3 = r_1 r_2 n + s_1 a r_2 + r_1 s_2 b \)

\( s_3 = s_1 s_2 \)

Hence \( \exists \ r, s \in \mathbb{Z} \), \( r n + s (a b) = 1 \)

Therefore since \( n \) is prime, and using claim 2, \( n \nmid a b \), as req'd.

\[ 0 = 2 \]
\[ 0 \times 3 = 2 \times 4 \]
\[ 3 = 4 \]

There is not a largest integer.

Prove True.

- To show True show that it could not not be True.
- Show its negative is False.
There is a largest integer.

\[ \exists n \in \mathbb{Z}, \forall m \in \mathbb{Z}, \, m \leq n \]

Let \( n \) be the largest integer. Now consider \( m = n + 1 \). Now \( m \in \mathbb{Z} \), and \( m > n \).

And so \( \neg (m \leq n) \).

Hence \( \forall m \in \mathbb{Z}, \, m \leq n \) is False

Hence \( \exists n \in \mathbb{Z}, \forall m \in \mathbb{Z}, \, m \leq n \) is False

or \( \neg (\exists n \in \mathbb{Z}, \forall m \in \mathbb{Z}, \, m \leq n) \) is True

i.e. There is not a largest integer.

Proof by contradiction:

- Start with an assumption and show that this leads to something that cannot be true. So assumption must be false.

related to indirect proof:

- want to prove \( S \) is True
Assume \( \neg S \) is True.

Show that this leads to something False (contradiction).

Have shown
\[
\neg S \implies \text{False}
\]
is True

By contraposition
\[
\neg \text{False} \implies \neg \neg S
\]
is True

Or
\[
\text{True} \implies S
\]
is True

So \( S \) must be True.