• Last lecture!

You have come a long way... you know what

\[ \exists \epsilon_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \implies g(n) \leq cf(n) \]

means and know why one might write it.

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Term Test 2: aug 21/30 ~ 72%

• scanning ongoing
  • should get back on website Wed.

PS 3: marking ongoing ...

PS 4: sample solutions posted by morning of Mon Dec 10th.

PS 4 questions:
Final exam
  • covers whole term

  • Exam info on course website now
    • past exams

  • pg 1-2
  • pg 21

  • some questions may be straightforward . Some may be more challenging.

Advice:  • read through all Q at the start

  • don’t fixate on a problem - move on when stuck and return

  • start to become a morning person!

Graphs
  • $G = (V, E)$
Connectivity: Can I get there from here?

- \( u, v \in V \) are connected in \( G \) iff there is a path from \( u \) to \( v \) using distinct, adjacent vertices from \( V \)

- \( G = (V, E) \) is connected iff \( \forall u, v \in V \), \( u \) and \( v \) are connected in \( G \).

Today: What is the minimum number of edges in a connected graph with \( n \) vertices?

Intuition

Consider:

1. \( |V| = n \) 
2. No.
Q: Can I remove an edge and maintain connectivity?

How are the cases different?

- When can we remove edges and maintain connectivity?

- Graphs 1, 4 contain cycles.

\textit{Def.}: Let $G = (V, E)$ be a graph.

A cycle in $G$ is a sequence of vertices $u_0, u_1, u_2, \ldots, u_k$ satisfying the following conditions:

- $k \geq 3$
- $u_0 = u_k$ and all other vertices $u_i$ distinct from each other and $u_0$.
- Each consecutive pair of vertices
The length of a cycle $= \# @edges in cycle = \# @vertices in cycle$

**Theorem:** Let $G = (V, E)$ be a graph. Assume $G$ is connected and contains a cycle. Let $e$ be any edge that is in a cycle in $G$. Then the graph $G' = (V, E \setminus \{e\})$ obtained by removing $e$ from $G$ is still connected.

- We can always remove edges from connected that contain cycles and maintain connectivity.
- Looking for a min $\# @edges$.

Consider graphs with no cycles.

**Def:** A tree is a graph that is connected and has no cycles.

**Q:** Is a tree a minimally connected graph?

**Theorem:** Let $G = (V, E)$ be a connected graph. If $G$ does not have a cycle then there does not exist an edge $e$ in $G$.
such that $G' = (V, E \setminus \{e\})$ is connected.

**Theorem:** Let $G$ be a tree. Then removing any edge from $G$ disconnects the graph.

**Theorem:** Let $G = (V, E)$ be a tree. Then $|E| = |V| - 1$.

- Proof by induction!

- Let $n$ be the # of vertices in $G$.

**Proof:** For all $n \in \mathbb{N}^+$, let $G = (V, E)$,

$(G$ is a tree $\land |V| = n) \implies |E| = n-1$.

**Predicate $P(n)$:** $\forall n \in \mathbb{N}^+, P(n)$.

**Base Case:** $n = 1$.

In this case, $G$ has a single vertex and no edges. Then $|E| = 0$ and $n-1 = 0$ and so $P(1)$ is True.
**Inductive Step:** Let $k \in \mathbb{N}^+$ and assume $P(k)$

\[ \forall G = (V,E), (G \text{ is a tree } \land |V| = k) \implies |E| = k - 1 \]

We want to prove $P(k+1)$

\[ \forall G = (V,E), (G \text{ is a tree } \land |V| = k + 1) \implies |E| = k \]

Let $G = (V,E)$ be a tree with $|V| = k + 1$. We need to prove that $|E| = k$.

So we need to remove some $u$ from $G$.

Which one

Let $u \in V$ be a vertex with exactly one neighbour.

Let $G' = (V', E')$ be the graph obtained by removing $u$ and the edge involving $u$ from $G$.

Then $|V'| = |V| - 1 = k$ and $|E'| = |E| - 1 = (k + 1) - 1$
Since \( G \) is a tree, so is \( G' \), since removing an edge would not create a cycle. \( G' \) has \( k \) vertices. So by the induction hypothesis, \( G' \) has \( |E'| = k-1 \) edges. And so, \( |E| = k \), as expected.

- Every connected graph has a tree
- So constraint on \# edges in a tree gives a lower bound on the \# edges in a connected graph.

**Theorem:** Let \( G = (V, E) \) be a graph. If \( G \) is connected, then \( |E| \geq |V| - 1 \).