Write a description for graph $G$:

$$G = (V, E)$$

where

$$V = \{A, B, C, D, E\}$$

$$E = \{(A, B), (A, C), (A, D), (C, D)\}$$

undirected $\Rightarrow (A, B) = (B, A)$

- The number of vertices in $G$ $|V| = 5$
- The number of edges in $G$ $|E| = 4$

Can prove results about any graphs
Example: Let $G = (V, E)$ be an arbitrary graph.

Then $|E| \leq \frac{|V|(|V|-1)}{2}$.

How to prove?

Translate:

$\forall G = (V, E) \in G$, $|E| \leq \frac{|V|(|V|-1)}{2}$

Proof: Let $G = (V, E)$ be an arbitrary graph.

Each edge in $G$ consists of a pair of vertices from $V$, where order does not matter. The max. number of edges is the same as the number of subsets of size 2, which we know is $\frac{|V|(|V|-1)}{2}$.

And so, $|E| \leq \frac{|V|(|V|-1)}{2}$.

\[ \square \]

Beware of double counting: $(u_i, u_j)$ is $\neq (u_j, u_i)$.

Concepts:

- Can we get to vertex $u_j$ from $u_i$ using the given edges?
Need terminology to make concepts precise.

**Def.** Let \( G = (V, E) \) and let \( \mathbf{v}_1, \mathbf{v}_2 \in V \).

We say that \( \mathbf{v}_1, \mathbf{v}_2 \) are adjacent if and only if \( (\mathbf{v}_1, \mathbf{v}_2) \in E \).

(\( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are neighbors)

**Defn.** Let \( G = (V, E) \) and let \( \mathbf{u}, \mathbf{u}' \in V \).

A path between \( \mathbf{u} \) and \( \mathbf{u}' \) is a sequence of distinct vertices
\[ \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V \]

satisfying:

\[ \mathbf{v}_0 = \mathbf{u} \quad \text{and} \quad \mathbf{v}_k = \mathbf{u}' \quad \text{(end points)} \]

- each consecutive pair of vertices are adjacent:
  \[ (\mathbf{v}_0, \mathbf{v}_1) \in E, \ldots, (\mathbf{v}_i, \mathbf{v}_{i+1}) \in E \]
distinct:
- valid path
- invalid path

distinct:
- no loops in path

- length of a path: \# of edges in sequence

- path can have length 0 (allow \( u = u' \))
- can be more than one path between \( u \) \( u' \)
- can be paths of different lengths from \( u \) to \( u' \)
- can be no path from \( u \) to \( u' \)
- distance from \( u \) to \( u' \) is the length of the shortest path from \( u \) to \( u' \)

(if no path exists, distance is \( \infty \))

- we say that vertices \( u \) and \( u' \) are connected if and only if there is a path
we say that a graph is connected if and only if for all pairs of vertices \( u, v \in V \), \( u \) and \( v \) are connected.

\[ \forall u, v \in V, \; u, v \text{ are connected.} \]

Define the predicate:

\[ \text{conn} \left( G, u, v \right) \; : \; \text{"} u \text{ and } v \text{ are connected vertices in graph } G \text{"} \]

where \( G \in G, \; u, v \in V, \; u \neq v \).

**Facts:**

1. \[ \text{conn} \left( G, u, v \right) \Rightarrow \text{conn} \left( G, v, u \right) \]

   \[ \exists \text{ path } \; u \rightarrow v_0 \rightarrow \cdots \rightarrow v_k \rightarrow u \]

   "\( u \) is a path from \( v_0 \) to \( v_k \), \( v_k \) to \( v_0 \)"

2. \[ \forall u, v, w \in V, \]

   \[ \text{transitivity} \; \text{conn} \left( G, u, v \right) \land \text{conn} \left( G, v, w \right) \]

   \[ \Downarrow \; \text{conn} \left( G, u, w \right) \]
\[ \Rightarrow \text{conn}(G, u, w) \]

**Proof:**

\[ u \rightarrow a_0, a_1, a_2, \ldots, a_m \rightarrow \ast \rightarrow v \]

\[ v \rightarrow \ast \rightarrow w \rightarrow b_0, b_1, b_2, \ldots, b_n \]

Path from \( u \) to \( w \):

\[ a_0, a_1, a_2, \ldots, a_m, b_0, b_1, b_2, \ldots, b_n \]

(Not general)

**Case:** \( a \)'s and \( b \)'s are distinct

The path from \( u \) to \( w \) is

\[ a_0, a_1, a_2, \ldots, a_m, b_0, b_1, b_2, \ldots, b_n \]

\( a \)'s and \( b \)'s are not distinct.

Path:

\[ u \rightarrow a_0, a_1, a_2, \ldots, a_k, b_{r_1}, b_{r_2}, \ldots, b_n \rightarrow w \]
k smallest index s.t. \( a_k = b_i \)

for some \( i \in \text{set} \).

Next questions:

\[ G = (V, E) \]

1. Is there a \( M_1 \) s.t. if \( |E| > M_1 \),
   \( G \) is necessarily connected?

2. Is there a \( M_2 \) s.t. if \( |E| < M_2 \),
   \( G \) is necessarily not connected?

\[ \begin{array}{ccc}
  G \text{ is not connected} & G \text{ might be connected} & G \text{ is connected} \\
  \hline
  M_2 & \text{?} & |E| \\
  \end{array} \]

It turns out:

\[ M_1 = \frac{(1V(1-1)(1V1-2)/2}{2} \]

(See Course Notes)

\[ M_2 = ? \]

(next time)