Team Test 1
Thu Oct 11

LO101 EX 100 1.5 - 2^{30}

Reminder of Induction Principle
Simple

- to prove stmts of form: \( \forall n \in \mathbb{N}, P(n) \)
  - variant: \( \forall n \in \mathbb{N}, n \geq m \implies P(n) \)

- idea: prove a base case: \( P(0) \)
  - if can also prove \( P(0) \implies P(1) \) \( \therefore \) know \( P(0) \)
    - \( P(0) \implies P(1) \)
    - \( P(2) \)
    - \( \vdots \)
  - want to prove \( \forall k \in \mathbb{N}, P(k) \implies P(k+1) \)

all together:

\[ P(0) \land (\forall k \in \mathbb{N}, P(k) \implies P(k+1)) \]

\[ \implies (\forall n \in \mathbb{N}, P(n)) \]
Variant: \( P(m) \land (\forall k \in \mathbb{N}, (k \geq m \Rightarrow P(k))) \Rightarrow (\forall n \in \mathbb{N}, n \geq m \Rightarrow P(n)) \)

given an \( m \in \mathbb{N} \)

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Consider:

\[ k \in \mathbb{Z}, n = 2k - 1 \]

The sum of the first \( n \) odd natural numbers is a perfect square.

\[ \text{The sum} = a \cdot a \text{ for some } a \in \mathbb{N} \]

Translate:

\[ \forall n \in \mathbb{N}, \exists a \in \mathbb{N}, \sum_{i=1}^{n} (2i-1) = a^2 \]

Discuss:

<table>
<thead>
<tr>
<th>( n )</th>
<th>Sum of the first ( n ) odd natural numbers</th>
<th>( n^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1+3 = 4</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>1+3+5 = 9</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>1+3+5+7 = 16</td>
<td>16</td>
</tr>
</tbody>
</table>

\[ \therefore \text{ take } a = n \text{ and prove } \forall n \in \mathbb{N}, \sum_{i=1}^{n} (2i-1) = n^2 \]
Proof: Let \( P(n) : \sum_{i=1}^{n} (2i-1) = n^2 \), where \( n \in \mathbb{N} \)

Base case: Let \( n = 0 \). Need to prove \( P(0) \).

Since \( \sum_{i=1}^{n} (2i-1) = \sum_{i=1}^{0} (2i-1) \)

\[ = 0 \]

\[ = (0)^2 \]

\[ = 0^2 \]

We know \( P(0) \).

Inductive step: We need to prove \( \forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1) \).

Let \( k \) be an arbitrary natural number and assume \( P(k) \). Then we know

\[ \sum_{i=1}^{k} (2i-1) = k^2 \]

Now \( \sum_{i=1}^{k+1} (2i-1) = \left[ \sum_{i=1}^{k} (2i-1) \right] + (2(k+1)-1) \)

By the induction hypothesis

\[ = k^2 + 2(k+1) - 1 \]

\[ = k^2 + 2k + 1 \]
\text{goal} \implies (k+1)^2

and so \( P(k+1) \) follows
and \( \forall k \in \mathbb{N}, P(k) \implies P(k+1) \)

Hence, by the principle of simple induction,
\[
\forall n \in \mathbb{N}, \quad \sum_{i=1}^{n} (2i-1) = n^2 \quad \Box
\]

\text{Example \hspace{1em} Prove that}

\( \forall x, y \in \mathbb{Z}, \forall n \in \mathbb{N}, \ S \mid (x-y) \implies S \mid (x^n - y^n) \)

\text{Proof: \hspace{1em} Let } x, y \in \mathbb{Z}.

Prove \( \forall n \in \mathbb{N}, \ S \mid (x-y) \implies S \mid (x^n - y^n) \)

Define the predicate
\[
P(n) : "S \mid (x-y) \implies S \mid (x^n - y^n)" ,
\]

\( P(n; x, y) \) \hspace{1em} \text{where } n \in \mathbb{N}, \ x, y \in \mathbb{Z}.

Then prove \( \forall n \in \mathbb{N}, P(n) \) using simple induction:

\text{base case}
Let $n=0$. We need to prove $P(0)$.

\[ \text{i.e. need to prove } S \mid (x-y) \Rightarrow S \mid (x^0-y^0) \]

Assume $S \mid (x-y)$.

Hence $\exists k \in \mathbb{Z}, \ x-y = 5k$.

Does $S \mid (x^0-y^0)$?

For integers $z \in \mathbb{Z}$, $z^0 = 1$.

So $x^0-y^0 = 1-1 = 0$

and $= 5 \cdot k$, for $k = 0$

Hence $S \mid (x^0-y^0)$

and $P(0)$ is True.

Inductive Step:

Let $k \in \mathbb{N}$ and assume $P(k)$.

That is $S \mid (x^k-y^k)$

Want to prove $S \mid (x^0-y^0) = S \mid (x^{k+1}-y^{k+1})$

Assume $S \mid (x^0-y^0)$. Let $c \in \mathbb{Z}$ such that $(x-y) = 5c$.

by the inductive hypothesis, we know $S \mid (x^k-y^k)$.

Let $d \in \mathbb{Z}$, be such that $x^{k+1}-y^{k+1} = 5d$. 

We need to show \[ S \left( x^{k+1} y^{k+1} \right) \]

Now consider \[ x^{k+1} y^{k+1} \]

\[ = x \cdot x^k - y \cdot y^k \]

\[ = x(x^k - y^k) + xy^k - yy^k \]

\[ = x(x^k - y^k) + y^k(x - y) \]

\[ = x \cdot 5d + y^k 5c \]

\[ = 5e \quad \text{for } e = xd + y^k c \quad c \in \mathbb{Z}. \]

and so \( P(t+1) \)

\[ \square \]