Now that we’ve studied both mergesort and quicksort, we’ll take a look at the running times of these algorithms. To analyse the running time of recursive algorithms, we need to study two things:

- the running time of the non-recursive operations in the code
- the structure of the recursion, i.e., how many recursive calls we make, and on what “smaller” inputs

This is a new technique that we have not covered before, so this worksheet will focus on getting the pieces ready to do a complete analysis together in lecture.

1. Here is the implementation of the helper function _partition used by quicksort.

   ```python
def _partition(lst: List, pivot: Any) -> Tuple[List, List]:
    smaller, bigger = [], []
    for item in lst:
        if item <= pivot:
            smaller.append(item)
        else:
            bigger.append(item)
    return smaller, bigger
```

What is the Big-Oh running time of _partition in terms of \( n \), the length of its input list?

\[ O(n) \] - \( n \) iterations, and append takes constant time (1 step)

2. How would your answer change each item is inserted at the front of the lists instead (e.g., smaller.insert(0, item))?

3. Here is the implementation of the helper function _merge used by mergesort.

   ```python
def _merge(lst1: List, lst2: List) -> List:
    index1, index2 = 0, 0
    merged = []
    while index1 < len(lst1) and index2 < len(lst2):
        if lst1[index1] <= lst2[index2]:
            merged.append(lst1[index1])
            index1 += 1
        else:
            merged.append(lst2[index2])
            index2 += 1
    return merged + lst1[index1:] + lst2[index2:]
```

\[ O(n_1 + n_2) \]

Let \( n_1 \) be the length of \( l1 \) and \( n_2 \) be the length of \( l2 \). (/*Note that _merge **does** work correctly even when given lists of different lengths.*) What is the maximum number of loop iterations for _merge, in terms of \( n_1 \) and/or \( n_2 \)?

\[ n_1 + n_2 \] \( \text{index1} + \text{index2} \text{ increases by 1 at each iteration} \)

4. Now consider the loop body. Does any operation in the loop body have a running time that depends on \( n_1 \) and/or \( n_2 \)? You should be able to justify your answer for every single operation given in this code, from \( \leq \) to _merged.append(…)_ to _lst1[index1]_.


5. Now we'll turn our attention to the structure of the recursive calls themselves. Because mergesort always evenly divides its input list into two equal halves, that's the algorithm we'll start with.

```python
def mergesort(lst: List) -> List:
    if len(lst) < 2:
        return lst[:]
    else:
        mid = len(lst) // 2
        left_sorted = mergesort(lst[:mid])
        right_sorted = mergesort(lst[mid:]):
        return _merge(left_sorted, right_sorted)
```

(a) Suppose we call mergesort on a list of length 8. This initial call makes two recursive calls. What is the length of the input lists to these two calls?

(b) Each of the recursive calls in part (a) makes two new recursive calls. What is the length of the input lists for these new recursive calls, and how many of them are there in total?

(c) Finally, each of the recursive calls in part (b) makes two new recursive calls. What is the length of the input lists for these new recursive calls, and how many of them are there in total?

(d) Let's try to present this information in a slightly different way to make the pattern clear. The table below shows the input list length vs. number of calls to mergesort on lists of that length. We've filled out the first column for you (showing the initial call); complete the rest of the table.

<table>
<thead>
<tr>
<th>Input list length</th>
<th>8</th>
<th>4</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of mergesort calls</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

6. Now consider the quicksort algorithm.

```python
def quicksort(lst: List) -> List:
    if len(lst) < 2:
        return lst[:]
    else:
        pivot = lst[0]
        smaller, bigger = _partition(lst[1:], pivot)
        return quicksort(smaller) + [pivot] + quicksort(bigger)
```

Its recursive step also makes two recursive calls, but unlike mergesort, the input list lengths are not necessarily always the same size.

(a) Suppose we call quicksort([0, 10, 20, 30, 40, 50, 60, 70]). After the _partition line, what are the lengths of smaller and bigger?

(b) Now let's look at the recursive structure of this call. Assume that for every recursive call to quicksort, the first element of the input is always the smallest element of the list. Using this assumption, complete the following table, which shows the input list length vs. number of quicksort calls. (You'll need to add more columns.)

<table>
<thead>
<tr>
<th>Input list length</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of quicksort calls</td>
<td>1</td>
<td>i</td>
<td>i</td>
<td>i</td>
<td>i</td>
<td>i</td>
<td>i</td>
<td>i</td>
<td>i</td>
</tr>
</tbody>
</table>