Last time:
  • the stability of a numerical algorithm
  • the conditioning of a mathematical problem

A numerically stable algorithm + a well-conditioned problem

⇒ then you will compute the exact result of
  a slightly different problem

⇒ then the solution to the slightly different problem will be close
to the desired solution

⇒ Know results that you compute will be accurate

\[ \hat{x} \approx \bar{x} \]
\[ \text{with rel error} \quad \frac{\hat{x} - \bar{x}}{\bar{x}} \approx 8x \]
\[ \Rightarrow \hat{x} = \bar{x} (1 + 8x) \]
the rel. error made by approx. \( f(\tilde{x}) \) by \( f(x) \) is:
\[
\left| \frac{f(\tilde{x}) - f(x)}{f(x)} \right| = \kappa_f(x) |\delta x|
\]
where \( \kappa_f(x) = \left| \frac{x \cdot f'(x)}{f(x)} \right| \) = condition number for evaluating \( f(x) \)

\( \kappa \) small - problem well conditioned

\( \kappa \) large - ill-conditioned

we saw: \( \kappa_N(x) = \frac{1}{2} \) well cond. \( \forall x \)

\( \kappa_{\ln}(x) = \frac{1}{\ln(x)} \) so illcond. for \( x \) near 1

Example: Consider \( f(x) = \ln(x) \)

Observe \( \ln(0.999) \approx -1.0005 \times 10^{-3} \)

Consider \( \ln(0.999(1 + 0.0001)) = -9.0049 \times 10^{-4} \)
\[ \text{rel. change to \( \ln \)} \]
\[ = 0.0005 \times 10^{-3} \]
\[ = (1 - 0.0009996) \]
\[ = 0.9996 \times 0.01\% \]

- A small change in problem (argument to \( \ln \))
  lead to a large change in result
- So problem must be ill-conditioned

\[ \ln^{-1}(x) = \frac{1}{\ln(x)} \]
\[ x = 0.999 \]
\[ \ln^{-1}(0.999) = \frac{1}{\ln(0.999)} \]
\[ = 999.5 \]

- The significance of a small error in \( x \) can be magnified by a factor \( \times 1000 \).
- This is independent of algorithm used to compute log.

What if you really want \( \ln(x) \) for \( x \) near 1?

Define \( w \) to be such that
\[ x = 1 + w \]
\[ \ln(x) = \ln(1 + w) \]

Consider \( g(w) = \ln(1 + w) \)

\[ \chi_g(w) = \frac{w \cdot g'(w)}{g(w)} \]

\[ = w \cdot \frac{1}{1 + w} \]

\[ \ln(x) \text{ ill cond. for } x \to 1 \]

\[ \text{corresponds to } w \to 0 \]

\[ \lim_{w \to 0} \chi_g(w) = 1 \]

\[ \therefore \ln(1 + w) \text{ is well conditioned} \]

Conclusion: for \( x \) near 1,

- Compute \( w \) instead of \( (x-1) \)
- Then take \( \ln(x) = \log_{1p}(w) \cdot g'(w) \)

- man \( \log_{1p} \)
In the first lecture in the course, you used a truncated version of the Maclaurin series expansion
\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \]
to develop an algorithm for calculating \( e^x \). But you also saw that it is not a good algorithm in practice because it produced results like:
- \( e^{-30} = -1.5 \times 10^{-4} \) (wrong, because \( e^x > 0 \) for all \( x \))
- \( e^{-40} = 3.7 \) (wrong, because \( e^x < 1 \) for all \( x < 0 \))

In our implementation of the algorithm, we stopped summing the series when the partial sums had converged to 8 places.

What went wrong?

Let's look into this in more detail and calculate \( e^{-5.5} = 0.0040868 \ldots \)

Using the series, and 5 decimal digit arithmetic, we get the following:

<table>
<thead>
<tr>
<th>i</th>
<th>terms</th>
<th>partial sums</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>1</td>
<td>-5.5000</td>
<td>-4.5000</td>
</tr>
<tr>
<td>2</td>
<td>+15.125</td>
<td>+10.625</td>
</tr>
<tr>
<td>3</td>
<td>-27.730</td>
<td>-17.105</td>
</tr>
<tr>
<td>4</td>
<td>+38.129</td>
<td>+21.024</td>
</tr>
<tr>
<td>5</td>
<td>-41.942</td>
<td>-20.918</td>
</tr>
<tr>
<td>6</td>
<td>+38.446</td>
<td>+17.528</td>
</tr>
<tr>
<td>7</td>
<td>-30.208</td>
<td>-12.680</td>
</tr>
</tbody>
</table>

(Note that the terms are now decreasing, which is a necessary condition for the series to converge.)

"converged" partial sum \( \approx 0.0026363 \) not approx= \( 0.0040868 \) exact

Why did we calculate such a poor approximation to the true result?
How to avoid the cancellation?

How to find $e^{-5.5}$ accurately?

Note: $e^{-x} = \frac{1}{e^x}$

1. Compute $e^{5.5}$ using series
   - all terms in sum positive
     $\Rightarrow$ no cancellation

   $e^{5.5} \approx 244.69$

   $e^{-5.5} = \frac{1}{e^{5.5}}$

   $= \frac{1}{244.69}$

   $= 0.0040868$

   if $x < 0$
   return $1 / \exp(\text{abs}(x))$

   else
   return $\exp(x)$

   avoids cancellation!