Last time:

New problem:

Given a continuous \( f: \mathbb{R} \to \mathbb{R} \), find an \( x^* \) s.t. \( f(x^*) = 0 \).

Could be 0 roots, 1 root, or many roots.

Developed bisection method based on repeated application of IVT.

Summary of observations from application of bisection:

Good:

- Given an initial bracket of root, bisection method guaranteed to converge to a root (other alg. don't have this guarantee).

- Only requires evaluating \( f(x) \) (other alg. require derivatives of \( f \)).

- Can determine in advance # iterations.
required for given abs. error tolerance

easy to implement

bad:

- converges slowly and not uni.
- next \( m_i \) could be farther from \( x^* \) than \( m_{i-1} \)
  \[ \Rightarrow f(m_i) \text{ could be larger than } f(m_{i-1}) \]

- does not make use of all available info. only \( f(m) \) 1 bit.

- may be hard to find initial bracket

\[ y = f(x) \]

\( \frac{A}{B} \) easier \( \frac{A}{B} \) hard to find initial bracket

\[ \Rightarrow \text{explore other techniques} \]
Fixed Point Iterations

- find an equivalent but different problem to solve
  has same solution.
- find $x^*$ s.t. $f(x^*) = 0$.
- define $g(x) = x - f(x)$
  
  now find $p^*$ that are fixed points
  do $g(x)$ i.e. $g(p^*) = p^*$

  $g(p^*) = p^*$
  $= p^* - [f(p^*)] = 0$ as needed

- fixed points of $g(x)$ are roots of $f(x)$

 fixed points correspond to values where $g(x)$
How to find fixed points of $g(x)$

Functional / fixed point iteration

- $x_0$ given
- $x_{i+1} = g(x_i)$, $i = 0, 1, 2, \ldots$

If this iteration converges, it converges to a fixed point of $g$.

Example: $f(x) = x - 0.2 \sin(x) - 0.5$

[recall: bisection found $x^* = 0.6154685$ in 21 steps]

How to convert to a fixed point problem?

Define $g(x) = 0.2 \sin(x) + 0.5$

So $f(x) = x - g(x)$

Consider: $x_0 = 0$; $i = 0$
repeat
\[ g(x) = 0.2 \sin(x) + 0.5 \]

\[ X_{in} = g(x_i), i = 0, 1, 2, \ldots \]

until \[ |X_{in} - x_i| < 5 \times 10^{-7} \]

\[ \Delta x_i < |x_i - x_{i-1}| \]

**results:**

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i )</th>
<th>( \Delta x_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5.0 \times 10^{-1}</td>
<td>5.0 \times 10^{-1}</td>
</tr>
<tr>
<td>2</td>
<td>5.95 \times 10^{-1}</td>
<td>9.6 \times 10^{-2}</td>
</tr>
<tr>
<td>3</td>
<td>6.12 \times 10^{-1}</td>
<td>1.6 \times 10^{-2}</td>
</tr>
<tr>
<td>4</td>
<td>6.149 \times 10^{-1}</td>
<td>2.7 \times 10^{-3}</td>
</tr>
<tr>
<td>5</td>
<td>6.1538 \times 10^{-1}</td>
<td>4.4 \times 10^{-4}</td>
</tr>
<tr>
<td>6</td>
<td>6.154618 \times 10^{-1}</td>
<td>1.9 \times 10^{-6}</td>
</tr>
<tr>
<td>7</td>
<td>6.154681 \times 10^{-1}</td>
<td>3.1 \times 10^{-7}</td>
</tr>
</tbody>
</table>

Converged in 9 iterations (bisection: 21)

repeat with different \( x_0 \) (starting point)

\( x_0 = 1.0 \) get convergence in 9 steps

-1.0
-10.0
-100.0
100.0
-1000.0

In this case, # steps not really dependent on \( x_0 \)
example: find the roots of

\[ f(x) = x^3 - x - 1 \]

using fixed point iteration

\[ g(x) = x^3 - 1 \]

for \( g(x) = x^3 - 1 \)

Try functional iteration on \( g(x) \)

Start with \( x_0 = 1 \)

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i )</th>
<th>( \Delta x_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.0</td>
<td>( 1.0 \times 10^{-6} )</td>
</tr>
<tr>
<td>2</td>
<td>-1.0</td>
<td>( 1.0 \times 10^{-6} )</td>
</tr>
<tr>
<td>i</td>
<td>( x_i )</td>
<td>( \Delta x )</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>7 ( \times 10^0 )</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1.8 ( \times 10^5 )</td>
<td></td>
</tr>
</tbody>
</table>

Try a different \( x_0 \).

\[ x_0 = 2 \]

Try a different \( g(x) \)

\[ f(x) = x^3 - (x+1) = 0 \]

\[ x^3 = (x+1) \]

\[ x = (x+1)^{\frac{1}{3}} \]