Tue Mar 6, 2018

- want to find roots of $f(x) = 0$

  by instead finding fixed points $x^*$

  of some $g(x)$

  i.e. find $x^*$ s.t. $g(x^*) = x^*$

fixed point iteration:

  \[
  \begin{cases}
  x_0 \text{ given} \\
  x_{i+1} = g(x_i) \quad i=0,1,2,...
  \end{cases}
  \]

- some iterations converge, others do not

- **Fixed Point Theorem**

  \[
  \begin{cases}
  g \in C[a,b] \quad \text{(continuous)} \\
  g: [a,b] \rightarrow [a,b] \\
  g \text{ is differentiable on } (a,b) \\
  \exists k \in \mathbb{R}, \forall x \in (a,b), |g'(x)| \leq k < 1
  \end{cases}
  \]

  \[
  \Rightarrow \quad \exists \text{ converges to a unique fixed point in } [a,b].
  \]

How quickly does \( \bigcirc \) converge?

If the conditions of Fixed Point Th. satisfied:
\[ 0 \quad |x_{i+1} - x_i| \leq k |x_i - x_{i-1}| \]

where \( k = \max \left| g'(x) \right| < 1 \quad x \in [a,b] \)

1. Distance between two iterates.

2. (a) \( |x_i - x^*| \leq k^i \max \{|x_0 - a|, |x_0 - b|\} \)
   
   (b) \( |x_i - x^*| \leq \frac{k^i |x_i - x_0|}{1 - k} \)

   distance from \( x_i \) to fixed point/root.

Apply to \( g(x) = 0.2 \sin(x) + 0.5 \)

with \( x_0 = 0 \)

\([a,b] = [-1, 1]\)

here \( k = 0.2 \)

Suppose we want \( |x_i - x^*| \leq \text{TOL} = 5 \times 10^{-7} \)

\[ z \quad \text{will get defined even if do } N \quad \text{iterations} \quad N \text{ s.t.} \]

\[ \text{if } (0.2^N \max \{|0-1|, |0-1|\}) \leq 5 \times 10^{-7} \]

\[ \text{then } |x_i - x^*| \leq 5 \times 10^{-7} \]
Solve \((0.2)^N \leq 5 \times 10^{-7}\)

or \(N > 9.014\)  \(\)  \(\text{\textit{(logs)}}\)

\(\therefore\) guaranteed convergence to TOL after 10 iterations.

\[\text{In practice, observed convergence to TOL in 9 iterations - okay since theory gives an overest. of \# req'd}.\]

\(r\) is a bound on derivative

Apply 2 (b) \(\quad x_0 = 0\) then \(x_1 = 0.5\)

\(L_0\) tells us that need \(N\) iterations \(\quad N \approx 3.7\) \(\quad |x_1 - x_0| \cdot (0.2)^N \leq 5 \times 10^{-7}\)

\(\quad \frac{1}{1 - 0.2}\)

\(\quad 0.5 \cdot \frac{(0.2)^N}{0.8} \leq 5 \times 10^{-7}\)

get \(N > 9.15 \ldots \quad N = 10\) or more

Summary notes on fixed point iterations:

good: \(\quad\) observed to be faster than bisection

\(\quad\) can

\(\quad\) math Theorems describe behavior
bad: need to construct a nice \( g(x) \) function - not all converge
if \( g'(x) \approx 1 \) near fixed point
corvergence can be slow.

Back to root finding:

approx. \( f(x) \) near \( x = x_0 \) by the str. line
that goes through \( (x_0, f(x_0)) \) that has same
slope as \( f \) at \( x_0 \).

eq str. line:
\[
\begin{align*}
y - f(x_0) &= f'(x_0)(x - x_0) \\
\end{align*}
\]
find root of this line set \( y = 0 \), solve for \( x \)

\[
x = x_0 - \frac{f(x_0)}{f'(x_0)}
\]
Repeat: to $x_2$, etc.

Iterative method

\[
\begin{cases} 
X_0 \text{ given guess} \\
X_{i+1} = X_i - \frac{f(x_i)}{f'(x_i)} \\
i = 0, 1, 2, \ldots
\end{cases}
\]

Newton's method.

A fixed point method with

\[
g(x) = x - \frac{f(x)}{f'(x)}
\]

Example: Apply Newton's method to

\[
f(x) = x - 0.2 \sin(x) - 0.5
\]

with $X_0 = 0$ until $|X_{i+1} - X_i| < 5 \times 10^{-7}$

[Recall: bisecion $[L, R] = [0, 1]$ 21 iterations
func. iteration obvious $g(x)$ 9 iterations]

have

\[
X_{i+1} = X_i - \frac{f(x_i)}{f'(x_i)}
\]
\[ \Delta x_i = |x_i - x_{i-1}| \]

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<th>( \Delta x_i )</th>
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Note: met the convergence criterion in 4 steps (almost in 3)
- change in \( x \) decreases by square factor each step.
  (quadratic convergence)
- N.m. can converge quickly

but:
- won't always converge
- requires working with \( f'(x) \)
  (which may not exist)
  \( \Rightarrow \) derivative free methods
• Find the equation of line through \((x_0, f(x_0))\), \((x_i, f(x_i))\) and then find its x-intercept and repeat.

• Equation: \(y - f(x_i) = \frac{f(x_i) - f(x_0)}{x_i - x_0} (x - x_i)\)

Set \(y = 0\), solve for \(x\).

\[ x = x_i - \frac{f(x_i)(x_i - x_0)}{f(x_i) - f(x_0)} \]

gives scheme:

\[
\begin{cases}
  x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}
\end{cases}
\]
\( \text{Let } x_0, x_1 \text{ be given } \quad i = 1, 2, \ldots \)

**Secant method**

\( \text{a str. line that cuts a curve at 2 points} \)

**Example** Use the secant method to compute a root of

\[ f(x) = x - 0.25 \sin(x) - 0.5 \]

Starting with \( x_0 = 0, x_1 = 1 \)

and stop when \( |x_{i+1} - x_i| < 5 \times 10^{-7} \)

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<td>4.2 \times 10^{-7}</td>
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Fast

- nice convergence like Newton's method
- no derr. needed
Apply Newton's method to find a root of
\[ f(x) = x - \tan(x) \text{ in } [3.5, 4.5] \]

\[ f(3.5) = 3.13 \quad f(4.5) = 0.137 \]

and \( f(x) \) is cts. in \([3.5, 4.5]\)

\[ : \text{ by INT, there is a root to find} \]

\[ f'(x) = 1 - \sec^2(x) \]
\[ = \frac{\cos^2(x) - 1}{\cos^2(x)} \quad \text{N.m.} \]
\[ = -\frac{\sin^2(x)}{\cos^2(x)} \]
\[ = -\tan^2(x) \]

\begin{center}
\begin{tabular}{c|c|c|c}
\text{\( i \)} & \( x_i \) & \( f(x_i) \) & \( f'(x_i) \) \\
\hline
0 & 4.0 & 2.84 & -1.34 \\
1 & 6.12 & 6.28 & -0.027 \\
2 & 238. & 238 & -0.137 \\
3 & 1978. & \ldots & \\
\end{tabular}
\end{center}

Not converging to root between 3.5 + 4.5

\[ f(x) \]

\[ 4.4934 \]

\[ x \]
• Started at an \( x \) where \( f'(x) \) too small pushed away from root.

• Try more values.
Newton’s method applied to:

\[ y = f(x) = x - \tan(x) \]
### Table 1

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Note: different initial guess
  - may converge / diverge
  - # iterations vary.

  - slow convergence near x=0

How to explain cycle vs. divergence:
  think of
  as a fixed pt. alg.

\[ g(x) = x - \frac{f(x)}{f'(x)} \]

  - f.p. theorem says look at \( g'(x) \)

\[ g'(x) = 1 - \frac{\left( \frac{f'(x)}{f''(x)} - \frac{f(x)f''(x)}{(f'(x))^2} \right)}{(f'(x))^2} \]

\[ = \frac{f(x)f''(x)}{(f'(x))^2} \]

\[ |g'(x)| < 1 \quad \text{provided} \quad \left| \frac{f(x)f''(x)}{(f'(x))^2} \right| < 1 \]
so N.m will converge when

\[
\left| \frac{f(x) f''(x)}{(f'(x))^2} \right| < 1
\]

Measuring Rate of Convergence

sequence: \( x_0, x_1, x_2, x_3, \ldots \)

that converges to \( x^* \).

Let \( e_i \) be a bound on \( |x_i - x^*| \)

abs. error in \( x_i \) term.

The rate of convergence \( r \) of the sequence to \( x^* \) is the largest value \( r \) s.t.

\[
\lim_{x_i \to x^*} \frac{e_{i+1}}{(e_i)^r} = c \neq 0
\]

\( e \) constant.

e.g. bisection: \( e_{i+1} = \frac{1}{2} e_i \)

so

\[
\lim_{x_i \to x^*} \frac{e_{i+1}}{e_i} = \frac{1}{2}
\]
So convergence rate for bisection is $r = 1$.

- If $r = 1$: linear convergence
- If $r > 1$: superlinear convergence
- If $r = 2$: quadratic convergence

Newton’s method + convergence

$$0 = f(x^*)$$
$$= f(x_i + (x^* - x_i))$$
$$= f(x_i) + f'(x_i)(x^* - x_i) + \frac{1}{2} f''(\Theta_i)(x^* - x_i)^2$$

for some $\Theta_i$ between $x_i$ and $x^*$

\[ \therefore \text{by } f'(x_i) \]

$$0 = \frac{f(x_i)}{f'(x_i)} + (x^* - x_i) + \frac{1}{2} \frac{f''(\Theta_i)(x^* - x_i)^2}{f'(x_i)}$$

$$x^* - (x_i - \frac{f(x_i)}{f'(x_i)}) = -\frac{1}{2} \frac{f''(\Theta_i)(x^* - x_i)^2}{f'(x_i)}$$
\[
\begin{align*}
\dot{x}_i - x_{i+1} &= -\frac{1}{2} \frac{\frac{f''(x)}{f'(x)}}{f'(x)} (x^* - x_i)^2 \\
e_{i+1} &= \left| \frac{-f''(x_i)}{2f'(x_i)} \right| e_i^2
\end{align*}
\]

\[\therefore \text{ if the method converges,}\]
\[\lim_{x_i \to x^*} \frac{e_{i+1}}{e_i^2} = \left| \frac{-f''(x^*)}{2f'(x^*)} \right|\]

\[r=2, \text{ N.m. has quadratic convergence.}\]
\[\text{when} \quad \frac{f''(x)}{f'(x)} \text{ is constant} \pm 0\]

Secant method

\[\text{can show} \quad \lim_{x_i \to x^*} \frac{e_{i+1}}{e_i e_i^{-1}} = c \pm 0\]

\[\text{and} \quad \lim_{x_i \to x^*} \frac{e_{i+1}}{e_i^r} = c \pm 0\]

\[r = \frac{1 + \sqrt{5}}{2} = 1.618\]