CSC236 Week 6, Tutorial 4 notes

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Derive a closed form for the following recurrence using the method of repeated substitution

$$f(n) = \begin{cases} 2 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 3f(n-1) - 2f(n-2) & \text{if } n > 1 \end{cases}$$

1 step 1: unwinding

I mentioned in tutorial that unwinding can be thought of as expanding the expression (by "digging down" the recursion), then simplifying (collapsing) it:

$f(n) = 3f(n-1) - 2f(n-2)$ *simplified*
$f(n) = 3[3f(n-2) - 2f(n-3)] - 2f(n-2)$ *expanding*
$f(n) = 9f(n-2) - 6f(n-3) - 2f(n-2)$ *simplifying*
$f(n) = 7f(n-2) - 6f(n-3)$ *simplified*
$f(n) = 7[3f(n-3) - 2f(n-4)] - 6f(n-3)$ *expanding*
$f(n) = 21f(n-3) - 14f(n-4) - 6f(n-3)$ *simplifying*
$f(n) = 15f(n-3) - 14f(n-4)$ *simplified*
...

2 step 2: general form

Let's have a look at those simplified guys:

$f(n) = 3f(n-1) - 2f(n-2)$ *simplified*
$f(n) = 7f(n-2) - 6f(n-3)$ *simplified*
$f(n) = 15f(n-3) - 14f(n-4)$ *simplified*

What can we predict about the next one?
Well the factor of the first term seems to always be a power of 2 minus 1.

$f(n) = [2^2 - 1]f(n-1) - 2f(n-2)$
$f(n) = [2^3 - 1]f(n-2) - 6f(n-3)$
$f(n) = [2^4 - 1]f(n-3) - 14f(n-4)$

And you get the drift: the factor of the second term is a power of 2 minus 2. So, without "unwinding" I can predict that the next *simplified* form will be:

$f(n) = [2^5 - 1]f(n-4) - [2^5 - 2]f(n-5)$
$f(n) = 31f(n-4) - 30f(n-5)$

Sweet. This means there is (seemingly) no point to "unwind" any further, we can conjecture that:

$f(n) = [2^{i+1} - 1]f(n-i) - [2^{i+1} - 2]f(n-i-1)$ *general form after some iterations*

3 step 3: clever substitution(s)

Now, we want to get rid of the $f(n - \texttt{bla})$ guys inside the expression of f. The only way to do this without magic is to get down to the base case(s). So let's do some "clever" substitutions:

$f(n) = [2^{i+1} - 1]f(n-i) - [2^{i+1} - 2]f(n-i-1)$ *I uncleverly chose a bad i, we need a better one*
$f(n) = [2^i - 1]f(n-i+1) - [2^i - 2]f(n-i+1-1)$ *substitute $i-1$ for i*
$f(n) = [2^i - 1]f(n-i+1) - [2^i - 2]f(n-i)$ *now we can get our base cases*
$f(i) = [2^i - 1]f(i-i+1) - [2^i - 2]f(i-i)$ *substitute i for n*
$f(i) = [2^i - 1]f(1) - [2^i - 2]f(0)$
$f(i) = [2^i - 1]3 - [2^i - 2]2$
$f(i) = (3)2^i - 3 - (2)2^i + 4$
$f(i) = 2^i + 1$

In this case we are done because the last substitution is pretty obvious (substitute n for i):

$f(n) = 2^n + 1$

4 extra step: sanity check

Let's check if our closed form works:

$f(2) = 2^2 + 1 = 5$
$f(3) = 2^3 + 1 = 9$
$f(4) = 2^4 + 1 = 17$

Check that this corresponds to the values the original form gives you and you are done.