CSC236 Week 4, Tutorial 3 notes

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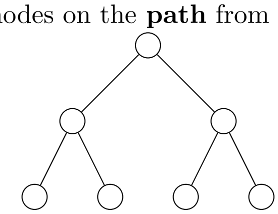
Prove that any full binary tree with more than zero nodes has exactly one more leaf than internal nodes.

1 Special Case with Simple Induction

With simple induction, it would be harder to prove the general case, but we can proove a special case easily. Consider a "total" full binary tree as a binary tree such that all leaf nodes are at the same distance from the root (there are the same number of nodes on the **path** from the root to any leaf). e.g.



The **height** of a tree is the longest **path** from its root to one of its leaves. The height of the singleton tree (tree with a single root node) is 0, because there are 0 nodes between the root and the root (as a leaf of the singleton tree). The height of the previous tree is 2. With simple induction we can prove a stronger statement about Total FBTs:

Call $P(h)$ the predicate: "A TFBT of height h has $2^h - 1$ internal nodes and 2^h leaves"

Proof. Simple induction
Basis:
The singleton tree, i.e.

has no internal nodes (an internal node is a node with at least one child) and 1 leaf.
Thus P(0) ($2^0 - 1 = 0$, $2^0 = 1$).

Inductive Reasoning:

Assume h, a natural number such that P(h).

Then a TFBT of height h has $2^h - 1$ internal nodes and 2^h leaves.
To get a TFBT of height $h + 1$, we add 2 children to every leaf. So we have twice as many leaves $2 * 2^h = 2^{h+1}$ and the new number of internal nodes is thus $2^h - 1 + 2^h = 2^{h+1} - 1$
Then P(h+1)

Then $P(h) \Rightarrow P(h + 1)$.

Conclusion:

By Simple Induction: $\forall h \in \mathbb{N}, P(h)$

$\square$

2 General case with Complete Induction

For every natural number h, there is only one TFBT of height h, so we can use simple induction. But FBTs are more complex, there are several different FBTs of height h. So **complete induction** is more straightforward.
Ok, this was helpful to get intuition and learn a bit about trees. Let's prove the general case with complete induction.

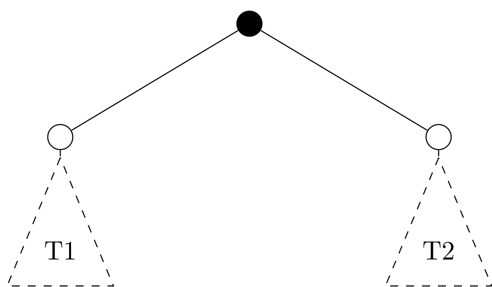
Call $P(h)$ the predicate: "a FBT of height h has exactly one more leaf than internal nodes"

Proof. Complete induction
Inductive Reasoning:

Assume $h \in \mathbb{N}$ and $\forall i \in \mathbb{N}, i < h \Rightarrow P(i)$

- case: $h = 0$ done previously (same "base case" for special and general cases). Thus $P(h)$.
- case: $h > 0$

  Call T a tree of size h. Since $h \geq 1$, T has 2 children, shown below as T1 and T2 (note that T1 need not be of the same height as T2):



  Given that T has height h, the height h_1 of T1 is bounded by: $h - 1 \geq h_1 \geq 0$ (same for the height of T2). By definition of an FBT, its children are also FBTs (of height strictly less than h). Thus, by the induction hypothesis, T1 has one more leaf node than internal nodes, and so does T2. T has as many more leaves than internal nodes than T1 and T2 do put together (i.e 2 more) but has one extra internal node (its root node, shown in black), thus it has one more leaf than internal nodes.
  Thus $P(h)$.

Then $\forall h \in \mathbb{N}, [\forall i \in \mathbb{N}, i < h \Rightarrow P(i)] => P(h)$

Conclusion:

By Complete Induction: $\forall h \in \mathbb{N}, P(h)$

□