CSC236 Week 3, Tutorial 2 notes

Alex Francois-Nienaber

Use a variation of simple induction to prove that for most natural numbers n, any set of n elements has 2^{n-1} subsets with an odd number of elements

1 step 1: define the predicate

Call P(n) the predicate: "any set of n elements has 2^{n-1} subsets with an odd number of elements"

Can we write this more formally? There is already notation for the set of all subsets of a set S, a.k.a. the powerset of S: $\mathcal{P}(\mathcal{S})$. Let's call $\mathcal{P}_{odd}(S)$ the set of all subsets (of a set S) containing an odd number of elements. Let's also define $\mathcal{P}_{even}(S)$ to be the set of all subsets (of a set S) containing an even number of elements (it might come in handy in the proof). Notice that for a given S, we have $\mathcal{P}_{odd}(S) \cup \mathcal{P}_{even}(S) = \mathcal{P}(\mathcal{S})$ and $\mathcal{P}_{odd}(S) \cap \mathcal{P}_{even}(S) = \emptyset$.

Now we can define P(n) as the predicate: "$\forall S, |S| = n \Rightarrow |\mathcal{P}_{odd}(S)| = 2^{n-1}$"

2 step 2: convince yourself it will hold for most n

Does P(0) hold? $2^{0-1} = 2^{-1} = 1/2$. A set cannot have a "half-element" so P(0) is nonsense.

Does P(1) hold? $\mathcal{P}(\{\pi\}) = \{\emptyset, \pi\}$; so $\mathcal{P}_{odd}(\{\pi\}) = \{\pi\}$ (the empty set has 0 elements and 0 is even). And $2^{1-1} = 2^0 = 1$. So it seems that P(1) does hold.

Check it for P(2) and so on until you are convinced.

3 step 3: prove it

Now that the preliminary work is done, let's use simple induction to prove P(n) for all n except 0.

Call $P(n)$ the predicate: "$\forall S, |S| = n \Rightarrow |\mathcal{P}_{odd}(S)| = 2^{n-1}$"

Proof. Simple induction
Basis:

Assume S, such that $|S| = 1$. Call π its only element, thus $S = \{\pi\}$

Then $\mathcal{P}(S) = \{\emptyset, \pi\}$
Then $\mathcal{P}_{odd}(S) = \{\pi\}$
Then $|\mathcal{P}_{odd}(S)| = 1 = 2^0 = 2^{1-1}$

Then $\forall S, |S| = 1 \Rightarrow |\mathcal{P}_{odd}(S)| = 2^{1-1}$
Thus P(1)

Inductive Reasoning:

Assume n, a natural number greater than 0 such that P(n).

Assume S such that $|S| = n + 1$. Call X a subset of S and π an element of S such that $S = X \cup \{\pi\}$ with $\pi \notin X$.

Then by the induction hypothesis: $|\mathcal{P}_{odd}(X)| = 2^{n-1}$
Recall from lecture that $|\mathcal{P}(X)| = 2^n$. And, since by definition $\mathcal{P}(X) = \mathcal{P}_{odd}(X) \cup \mathcal{P}_{even}(X)$, we have $|\mathcal{P}_{even}(X)| = 2^{n-1}$
Also from lecture: $\mathcal{P}(X \cup \{\pi\}) = \mathcal{P}(X) \cup \{s \cup \{\pi\} | s \in \mathcal{P}(X)\}$
($\{s \cup \{\pi\} | s \in \mathcal{P}(X)\}$ is what I called NEWSTUFF in tutorial).
So we have:

$\mathcal{P}(S) = \mathcal{P}(X) \cup \{s \cup \{\pi\} | s \in \mathcal{P}(X)\}$
$\mathcal{P}(S) = \mathcal{P}_{odd}(X) \cup \mathcal{P}_{even}(X) \cup \{s \cup \{\pi\} | s \in \mathcal{P}(X)\}$
$\mathcal{P}(S) = \mathcal{P}_{odd}(X) \cup \mathcal{P}_{even}(X) \cup \{s \cup \{\pi\} | s \in \mathcal{P}_{odd}(X)\} \cup \{s \cup \{\pi\} | s \in \mathcal{P}_{even}(X)\}$

We have $|\mathcal{P}_{odd}(X)| = |\{s \cup \{\pi\} | s \in \mathcal{P}_{odd}(X)\}| = 2^{n-1}$ (*since we are adding the element π to every set in $P_{odd}(X)$ so it doesn't change the number of sets in* $\{s \cup \{\pi\} | s \in \mathcal{P}_{odd}(X)\}$). Same argument for $\mathcal{P}_{even}(X)$.

Finally $\{s \cup \{\pi\} | s \in \mathcal{P}_{odd}(X)\}$ contains **only even subsets**, and $\{s \cup \{\pi\} | s \in \mathcal{P}_{even}(X)\}$ contains **only odd subsets** because by adding a single element to a set, if it had an odd size it will have an even size and vice-versa. Hence:
$\mathcal{P}_{odd}(S) = \mathcal{P}_{odd}(X) \cup \{s \cup \{\pi\} | s \in \mathcal{P}_{even}(X)\}$

And since $\mathcal{P}_{odd}(X) \cap \{s \cup \{\pi\} | s \in \mathcal{P}_{even}(X)\} = \emptyset$, we have:
$|\mathcal{P}_{odd}(S)| = |\mathcal{P}_{odd}(X)| + |\{s \cup \{\pi\} | s \in \mathcal{P}_{even}(X)\}|$
$|\mathcal{P}_{odd}(S)| = 2^{n-1} + 2^{n-1} = 2^n$

Then P(n+1)

Then $P(n) \Rightarrow P(n+1)$.

Conclusion:

By Simple Induction: $\forall n \in \mathbb{N} - \{0\}, P(n)$

□