CSC 236 — Fall 2014
Assignment 2
Question 1 and 2 Solutions

1. Claim: $f(n) \leq 236 \cdot n^4$ for all positive natural numbers n.
   For $n \in \mathbb{N}$, let $P(n)$ be: $f(n) \leq 236 \cdot n^4$. The claim is: $\forall n \in \mathbb{N}, n \geq 1 \rightarrow P(n)$.

   Proof. By Complete Induction.
   Base Cases $P(1), P(2)$:
   $f(1) = 3f\left(\left\lfloor \frac{2 \cdot 1}{5} \right\rfloor\right) + 6 \cdot 1^4 = 3f(0) + 6 = 36 \leq 236 = 236 \cdot 1^4$.
   $f(2) = 3f\left(\left\lfloor \frac{2 \cdot 2}{5} \right\rfloor\right) + 6 \cdot 2^4 = 3f(0) + 6 \cdot 16 = 30 + 96 \leq 200 \leq 236 \leq 236 \cdot 2^4$.
   Inductive Step. Let $n \geq 3$:
   (IH) Assume that for every natural number k, if $1 \leq k < n$ then $P(k)$.
   Note that:

   (a) The second clause of the definition of f applies to $f(n)$, since $n \geq 3 \geq 1$.
   (b) The (IH) for $k = \left\lfloor \frac{2n}{5} \right\rfloor$ is: $f\left(\left\lfloor \frac{2n}{5} \right\rfloor\right) \leq 236 \cdot \left(\left\lfloor \frac{2n}{5} \right\rfloor\right)^4$.
   The (IH) actually applies for $\left\lfloor \frac{2n}{5} \right\rfloor$, since $1 \leq \left\lfloor \frac{2n}{5} \right\rfloor < n$ when $n \geq 3$:
      i. $1 = \left\lfloor \frac{2 \cdot 3}{5} \right\rfloor \leq \left\lfloor \frac{2n}{5} \right\rfloor$.
      ii. $\left\lfloor \frac{2n}{5} \right\rfloor = \left\lfloor n - \frac{3n}{5} \right\rfloor \leq \left\lfloor n - \frac{3 \cdot 3}{5} \right\rfloor \leq \lfloor n - 1 \rfloor = n - 1 < n$.

   Then $P(n)$ is true because:

$$
\begin{aligned}
f(n) &= 3f\left(\left\lfloor \frac{2n}{5} \right\rfloor\right) + 6n^4 \text{ [from definition, see note (a)]} \\
&\leq 3 \cdot 236 \cdot \left(\left\lfloor \frac{2n}{5} \right\rfloor\right)^4 + 6n^4 \text{ [from (IH), see note (b)]} \\
&\leq 3 \cdot 236 \cdot \left(\frac{2n}{5}\right)^4 + 6n^4 \\
&\leq 1000 \cdot \left(\frac{n}{2}\right)^4 + 6n^4 \\
&= \frac{1000}{16} \cdot n^4 + 6n^4 \\
&\leq 100n^4 + 6n^4 \\
&\leq 236n^4
\end{aligned}
$$

   □

2.

   (a) $P(236)$: $T(236) \leq T(237)$.
   $T(236) = 2 + T\left(\left\lfloor \frac{236}{2} \right\rfloor\right) + T\left(\left\lceil \frac{236}{2} \right\rceil\right) = 2 + T(118) + T(118)$.
   $T(237) = 2 + T\left(\left\lfloor \frac{237}{2} \right\rfloor\right) + T\left(\left\lceil \frac{237}{2} \right\rceil\right) = 2 + T(118) + T(119)$.
   Assuming $P(118)$ would prove $P(236)$.
   Suppose $P(118)$. Then $T(118) \leq T(119)$. So $T(236) = 2 + T(118) + T(118) \leq 2 + T(118) + T(119) = T(237)$.
   (b) $P(237)$: $T(237) \leq T(238)$.
   $T(237) = 2 + T\left(\left\lfloor \frac{237}{2} \right\rfloor\right) + T\left(\left\lceil \frac{237}{2} \right\rceil\right) = 2 + T(118) + T(119)$.
   $T(238) = 2 + T\left(\left\lfloor \frac{238}{2} \right\rfloor\right) + T\left(\left\lceil \frac{238}{2} \right\rceil\right) = 2 + T(119) + T(119)$.
   Assuming $P(118)$ would prove $P(237)$.
   Suppose $P(118)$. Then $T(118) \leq T(119)$. So $T(237) = 2 + T(118) + T(119) \leq 2 + T(119) + T(119) = T(238)$.

(c) Base Case $P(1)$: $T(1) = 3 \le 2+3+3 = 2+T(1)+T(1) = 2+T\left(\lfloor \frac{2}{2} \rfloor\right)+T\left(\lceil \frac{2}{2} \rceil\right) = T(2) = T(1+1)$.

Inductive Step. Let $n \ge 2$:

(IH) Assume that for every natural number k, if $1 \le k < n$ then $P(k)$.

Note that:

i. If n is even: $\frac{n}{2}$ is an integer.
Then since $n \ge 2$: $1 \le \frac{n}{2} < \frac{n}{2} + \frac{n}{2} = n$.
So by the (IH) for $\frac{n}{2}$: $T\left(\frac{n}{2}\right) \le T\left(\frac{n}{2}+1\right)$.

ii. If n is odd: $\frac{n-1}{2}$ is an integer.
And since also $n \ge 2$: $n \ge 3$.
Then since $n \ge 3$: $1 = \frac{3-1}{2} \le \frac{n-1}{2} < \frac{n-1}{2} + \frac{n+1}{2} = n$.
So by the (IH) for $\frac{n-1}{2}$: $T\left(\frac{n-1}{2}\right) \le T\left(\frac{n-1}{2}+1\right)$.

Now $P(n)$ is true because:

Case n is even:

Then $\frac{n}{2}$ is an integer.

So each of $\lfloor \frac{n}{2} \rfloor$, $\lceil \frac{n}{2} \rceil$, and $\lfloor \frac{n+1}{2} \rfloor$ (which is equal to $\lfloor \frac{n}{2} + \frac{1}{2} \rfloor$), are equal to $\frac{n}{2}$.

And $\lceil \frac{n+1}{2} \rceil = \lceil \frac{n}{2} + \frac{1}{2} \rceil = \frac{n}{2} + 1$.

So:

$$\begin{aligned}
T(n) &= 2+T\left(\left\lfloor \frac{n}{2} \right\rfloor\right)+T\left(\left\lceil \frac{n}{2} \right\rceil\right) \text{ [by definition of } T\text{]} \\
&= 2+T\left(\frac{n}{2}\right)+T\left(\frac{n}{2}\right) \\
&\le 2+T\left(\frac{n}{2}\right)+T\left(\frac{n}{2}+1\right) \text{ [from (IH), see note i]} \\
&= 2+T\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right)+T\left(\left\lceil \frac{n+1}{2} \right\rceil\right) \\
&= T(n+1) \text{ [by definition of } T\text{]}
\end{aligned}$$

Case n is odd:

Then $\frac{n-1}{2}$ is an integer.

So each of $\lceil \frac{n}{2} \rceil$ (which is equal to $\lceil \frac{n-1}{2} + \frac{1}{2} \rceil$), $\lfloor \frac{n+1}{2} \rfloor$ (which is equal to $\lfloor \frac{n-1}{2} + 1 \rfloor$), and $\lceil \frac{n+1}{2} \rceil$ (which is equal to $\lceil \frac{n-1}{2} + 1 \rceil$), are equal to $\frac{n-1}{2} + 1$.

And $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{n-1}{2} + \frac{1}{2} \rfloor = \frac{n-1}{2}$.

So:

$$\begin{aligned}
T(n) &= 2+T\left(\left\lfloor \frac{n}{2} \right\rfloor\right)+T\left(\left\lceil \frac{n}{2} \right\rceil\right) \text{ [by definition of } T\text{]} \\
&= 2+T\left(\frac{n-1}{2}\right)+T\left(\frac{n-1}{2}+1\right) \\
&\le 2+T\left(\frac{n-1}{2}+1\right)+T\left(\frac{n-1}{2}+1\right) \text{ [from (IH), see note ii]} \\
&= 2+T\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right)+T\left(\left\lceil \frac{n+1}{2} \right\rceil\right) \\
&= T(n+1) \text{ [by definition of } T\text{]}
\end{aligned}$$