CSC 236 — Fall 2014
Assignment 0

1. The heart of a proof of $\forall n \in \mathbb{N}, P(n)$ by Simple Induction is proving $(S0): \forall n \in \mathbb{N}, P(n) \to P(n+1)$.
   Let's start by contrasting $(S0)$ with the *syntactically* similar $(S1): \forall n \in \mathbb{N}, P(n) \to \forall n \in \mathbb{N}, P(n+1)$.

   (a) Write a symbolic statement equivalent to $\forall n \in \mathbb{N}, P(n+1)$, that doesn't use addition ("+"), by using an implication. Give a Natural English Statement of your statement's meaning.

   (b) $(S1)$ can be parenthesized in more than one way. Put in parentheses to show its most likely intended interpretation.

   (c) Is $(S1)$ true for every predicate P? If so, give a structured direct proof of $(S1)$, following the structure as originally given above (not modified by what you did in (a)), otherwise define a predicate P for which $(S1)$ is false (no proof required). Note: predicates can be defined by simply stating the values for which they are true.

   (d) State the converse of $(S1)$, and repeat (c) for it.

2. Now let's examine $(S0)$.

   (a) Write the negation of $(S0)$, in the form where negation is completely 'pushed inside' (i.e. where negation occurs only on named predicates).

   (b) Find three different Ps such that $(S0)$ is true, and show their truth tables (which are infinitely long, so show enough to get the idea, and mention how the table continues). Like with program test cases, think of the 'extreme/boundary' case Ps and 'typical/generic' case Ps: pick three Ps as distinct from each other as possible.

   (c) Show a truth table for a typical/generic P for which $(S0)$ is false.

   (d) $(S0)$ is only *part* of what needs to be proven to use Simple Induction: alone it doesn't (always) imply $\forall n \in \mathbb{N}, P(n)$. State (without proof) which, if any, of your Ps from (b) satisfy $\forall n \in \mathbb{N}, P(n)$.

   (e) The first four parts of this question should suggest that there's a statement of the form $\exists n \in \mathbb{N}, P(n) \to \exists [...]$, that is equivalent to $(S0)$ and captures the idea that unless P is always false there's a point up to which (maybe vacuosly) something is happening, and then afterwards something else is happening. Complete that equivalent symbolic statement.

3. The Principle of Simple Induction (SI) claims: $(P(0) \wedge \forall n \in \mathbb{N}, P(n) \to P(n+1)) \to \forall n \in \mathbb{N}, P(n)$. It's natural to ask whether the converse is true. State the converse and give a structured proof or disproof of it.

4. Give a structured proof of this partial version of the principle: $(S2): P(0) \wedge (S0) \to P(0) \wedge P(1) \wedge P(2)$. Don't assume (SI) in your proof.

5. To use the technique of Simple Induction for some P we prove $(S3): P(0) \wedge \forall n \in \mathbb{N}, P(n) \to P(n+1)$.

   (a) Write the negation $\neg(S3)$ in the form where negation is completely pushed inside, and give a Natural English Statement of its meaning.

   (b) Based on the main connective in your version of $\neg(S3)$ show two illustrative truth tables of Ps for which $(S3)$ is false.

6. Consider the sequence defined by:

$$
\begin{aligned}
a_1 &= 2 \\
a_n &= \left(a_{\lfloor \sqrt{n} \rfloor}\right)^2 + 3a_{\lfloor \sqrt{n} \rfloor}, \text{ for } n \geq 2.
\end{aligned}
$$

   (a) Write a Python function that takes a natural number $n \geq 1$ and returns a_n.

   (b) Write a Python function that takes a natural number n. If $n \geq 2$ it returns a_n, otherwise it raises an exception.
   Do not write any helper functions, nor call your function from (a).