CSC236 fall 2014, Assignment 1
sample solutions and notes

Due October 3rd, 10 p.m.

1. We've shown for $m = 3$ or 4 and most natural numbers n, that $m^n \geq n^m$. It's tedious to repeat this proof for all the natural numbers $m > 1$, so use induction on n to prove:

$$\forall m \in \mathbb{N}, \exists k \in \mathbb{N}, \forall n \in \mathbb{N}, (m > 1 \wedge n \geq k) \Rightarrow m^n \geq n^m$$

Hint #1: You are proving a claim with multiple quantifiers, but you only need induction for the innermost, $\forall n \in \mathbb{N}$.
Hint #2: You may find it useful to notice that

$$(n+1)^m = \left(\frac{n+1}{n}\right)^m \times n^m = (1 + 1/n)^m \times n^m$$

Sample Solution: After some experimentation, it looks as though with the exception of $m = 2$ we have $m^n \geq n^m$ provided $n \geq m$. So we can prove the result for $m = 2$ as a special case.

Also, as a tutorial exercise this week we proved that for $n \geq 3$, $(1 + 1/n)^n \leq n$, and this turns out to be useful. Put these together for a proof:

Proof (by induction on n): $\forall m \in \mathbb{N}, \exists k \in \mathbb{N}, \forall n \in \mathbb{N}, (m > 1 \wedge n \geq k) \Rightarrow m^n \geq n^m$.

Assume m is an arbitrary natural number. We define a predicate

$$P(n) : m^n \geq n^m$$

Case 0: assume $m = 2$: Pick $k = 4$, a natural number. We must show that for all natural numbers $n \geq 4$, $P(n)$ is true.

Base case: If $n = k = 4$, then $2^4 = 4^2 = n^4$, so $P(4)$ is true.

Induction step: Assume that n is a generic natural number greater than or equal to 4, and that $P(n)$ is true, that is $m^n \geq n^m$ (Induction Hypothesis). We must show that $P(n+1)$ follows from this.
Then

$$\begin{aligned}
2^{n+1} &= 2 \times 2^n \geq 2 \times n^2 && \text{\#by IH} \\
&\geq (1 + 1/4)^2 \times n^2 && \text{\#since } 2 > \frac{25}{16} \\
&\geq (1 + 1/n)^2 \times n^2 && \text{\#since } (1 + 1/4) \geq (1 + 1/n) \text{ if } n \geq 4 \\
&= \left(\frac{n+1}{n}\right)^2 \times n^2 = (n+1)^2
\end{aligned}$$

So, $P(n+1)$ follows from $P(n)$
Conclude that for all natural numbers $n \geq k = 4$, $m^n \geq n^m$, by the principle of induction.
Thus our claim holds when $m = 2$.

Case 1: assume $m > 2$: Pick $k = m$ and note that, by the result from tutorial exercise, for $m \geq 3$.

$$(1 + 1/m)^m \leq m$$

We need to show that for all natural numbers n greater than or equal to $k = m$, $P(n)$ is true. Assume n is an arbitrary natural number greater than or equal to m, and that $P(n)$ is true, that is $m^n \geq n^m$. We need to prove that $m^{n+1} \geq (n+1)^m$ follows from this.

Base case, $n = k = m$: $m^n = m^m \geq n^m$, so $P(k)$ is true.

Induction step: Assume n is an arbitrary natural number greater than or equal to m, and that $m^n \geq n^m$, that is $P(n)$ (Induction Hypothesis). We must show that $P(n+1)$ follows from this.

Then

$$\begin{aligned} m^{n+1} &= m \times m^n \geq m \times n^m && \#\text{by the Induction Hypothesis} \\ &\geq (1 + 1/m)^m \times n^m && \#\text{by tutorial exercise, } m \geq (1+1/m)^m \\ &\geq (1 + 1/n)^m \times n^m && \#\text{since } n \geq m, 1/m \geq 1/n \\ &= \left(\frac{n+1}{n}\right)^m \times n^m = (n+1)^m \end{aligned}$$

So $P(n+1)$ follows from $P(n)$

Conclude that for all natural numbers $n \geq m$, $m^n \geq n^m$, by the principle of induction. Thus our claim holds for an arbitrary $m \geq 3$.

Since our claim holds when $m = 2$ and $m \geq 2$, it holds for all natural numbers m greater than 1.

2. Consider the following equation:

$$(\sqrt{5} + 2)(\sqrt{5} - 2) = 1$$

You can complete parts (a) and (b) before you learn about the Principle of Well Ordering. Once you've learned the PWO, you can complete (c).

(a) Re-write the equation until you have an equation with $\sqrt{5}$ on the left and a ratio of two expressions involving $\sqrt{5}$ on the right.

Hint #1: Don't multiply out the bracketed expressions on the left — you'll lose the $\sqrt{5}$s.

Hint #2: There are two similar ways to do this, yielding different expressions on the right.

Sample solution: There are two like equations, depending on whether you divide 1 by $(\sqrt{5} - 2)$ or by $(\sqrt{5} + 2)$:

$$\sqrt{5} = \frac{5 - 2\sqrt{5}}{\sqrt{5} - 2} \qquad\qquad \sqrt{5} = \frac{5 + 2\sqrt{5}}{\sqrt{5} + 2}$$

There's no reason (yet) to prefer one equation over the other.

(b) Assume there are natural numbers m and n such that $\sqrt{5} = m/n$. Substitute m/n for $\sqrt{5}$ in your equation from (a). Simplify the ratio on the right hand side, to get a fraction of integers with the denominator a natural number less than n. If this doesn't work, go back to (a) and derive the other expression, then try that one.

Show your substitution and simplification, and explain why the denominator is less than n.

Sample solution: Substitute m/n, and then simply so the denominator is simple, yielding two possibilities:

$$\sqrt{5} = \frac{5 - 2\sqrt{5}}{\sqrt{5} - 2} \quad \text{becomes} \quad \frac{m}{n} = \frac{5 - 2(m/n)}{m/n - 2} = \frac{5n - 2m}{m - 2n}$$

$$\sqrt{5} = \frac{5 + 2\sqrt{5}}{\sqrt{5} + 2} \quad \text{becomes} \quad \frac{m}{n} = \frac{5 + 2(m/n)}{(m/n) + 2} = \frac{5n + 2m}{m + 2n}$$

Since $2^2 = 4 < 5 < 9 = 3^2$, we know that $2 < \sqrt{5} < 3$, so if $m/n = \sqrt{5}$, then $m = \sqrt{5}n$, in other words $2n < m < 3n$. Subtracting $2n$ from this inequality gives us:

$$0 < m - 2n < n$$

So, the denominator of the first equation is a natural number less than n. We'll use that one!

(c) Use the Principle of Well Ordering to derive a contradiction from the assumption in the previous part. What can you conclude?

Sample solution: In the previous part we assumed there were natural numbers m, n such that $m/n = \sqrt{5}$. A consequence of that assumption is that

$$R = \{n \in \mathbb{N} \mid \exists m \in \mathbb{N}, m/n = \sqrt{5}\}$$

...is not empty — it contains at least our assumed n. By the Principle of Well Ordering, since R is a non-empty subset of $\mathbb{N}$, it has a smallest element n_0, with a corresponding m_0 so that $m_0/n_0 = \sqrt{5}$. By the previous question we have

$$\frac{m_0}{n_0} = \frac{5n_0 - 2m_0}{m_0 - 2n_0}$$

Oops — $m_0 - 2_n 0$ is a natural number that qualifies for membership in R, since the numerator $5n_0 - 2m_0$ must be a non-negative integer (the denominator is, and m/n is positive). It is smaller than n_0, a contradiction. This means that our assumption that there are natural numbers m, n such that $m/n = \sqrt{5}$ is false. Indeed, $\sqrt{5}$ is irrational.

3. Read over, and experiment with, the Python function **is_b_list**:

```
def is_b_list(x):
    """(object) -> bool

    Return whether x is a binary list.

    >>> is_b_list("b_list")
    False
    >>> is_b_list(0)
    True
    >>> is_b_list([0, 0])
    True
    >>> is_b_list([[0]])
    False
    """
    return (x == 0 or
            (isinstance(x, list) and len(x) == 2
             and all([is_b_list(y) for y in x])))
```

Define the size of a binary list as the number of left brackets in its Python representation, i.e. the total number of list objects in the binary list. So 0 has size 0 and `[0, 0]` has size 1.

You can complete part (a) before we finish covering examples of induction. The particular Inductive Principle and associated proof form you'll rely on in (b) has a special name: "complete" induction. If you work on part (b) before we cover that principle you should try to convince yourself by inductive reasoning. The principle will then match your reasoning or help you complete your reasoning.

(a) Experiment until you find a formula (probably recursive) that computes the number of different binary lists of size s. Notice that if you call your formula **C(s)**, then **C(0)** computes 1 and **C(1)** also computes 1.

Sample solution: There is a single binary list of size 0: **0**.

There is also a single binary list of size 1: **[0, 0]**.

There are two binary lists of size 2: **[0, [0, 0]]** and **[[0, 0], 0]**.

There are 5 binary lists of size 3:

[0, [0, [0, 0]]] [0, [[0, 0], 0]]

[[0, 0], [0, 0]]

[[0, [0, 0]], 0] [[[0, 0], 0], 0]

They can be organized first by those with a binary list of size 0 at element 0 paired with a binary list of size 2 at element 1, then those with a binary list of size 1 at both elements 0 and 1, then those with a binary list of size 2 at element 0 paired with a binary list of size 0 at element 1. This suggests a general formula for computing the total number of binary lists of size s:

$$\mathsf{C(s)} = \begin{cases} 1 & \text{if } s = 0 \\ \sum_{i=0}^{s-1} C(i)C(s-i-1) & \text{otherwise} \end{cases}$$

(b) Use complete induction to prove that your **C(s)** correctly computes the number of different binary lists of size s

Sample solution: I need to prove $P(s)$: that for every natural number s, the formula **C(s)** correctly computes the number of different binary list sof size s

Assume that $P(k)$ is true for every natural number k less than s.

Case 0, assume $s = 0$: There is a single binary list of size 0, the binary list **0**, so **C(0) = 1** correctly computes the number of such lists.

Case 1, assume $s > 0$: A binary list of size $s > 0$ contains a binary list at index 0 of size k for some natural number $0 \leq k < s$, and a binary list at index 1 of size $s - 1 - k$, since the number of left brackets in their Python representation must add up to $s - 1$. For any particular k in $[0, s-1]$ there are $C(k) \times C(s-1-k)$ combinations of lists at indices 0 and 1, since $0 \leq k, s-1-k < s$ and our assumption is that $C(k)$ and $C(s-1-k)$ correctly compute the number of binary lists of these sizes. Summing over all configurations from lists at index 0 having size 0 to lists at index 0 having size $s - 1$ gives us

$$\mathsf{C(s)} = \begin{cases} 1 & \text{if } s = 0 \\ \sum_{i=0}^{s-1} C(i)C(s-i-1) & \text{otherwise} \end{cases}$$

In both possible cases, $P(n)$ follows from $P(k)$ for all $k \in \mathbb{N}, k < n$.

Since I was reasoning about a generic representative of the natural numbers, s, I conclude for all $s \in \mathbb{N}$, $P(s)$ follows from $\forall k \in \mathbb{N}, k < n \Rightarrow P(k)$.

So, by the principle of complete induction, $\forall s \in \mathbb{N}, P(s)$.