CSC 236 — Fall 2014
Assignment 0 — Process and Solutions
Version β.

1.

(a) $\forall n \in \mathbb{N}, P(n+1)$ is a universal. It claims $P(n+1)$ for $n = 0, 1, 2, 3, \ldots$, i.e. $P(0+1), P(1+1), P(2+1), P(3+1), \ldots$, i.e. $P(1) \wedge (2) \wedge P(3) \wedge P(4), \ldots$, i.e. that P is true for all natural numbers larger than zero, i.e. if n is a natural number greater than zero then $P(n)$, i.e $\forall n \in \mathbb{N}, n > 0 \rightarrow P(n)$. Either of the two prose statements in the previous sentence would be a fine Natural English Statement of this. To understand the symbolic statement it's especially important to be able to think of it in the form that doesn't mention the variable "n".

(b) There are two syntactically valid parsings: $\forall n \in \mathbb{N}, [P(n) \rightarrow \forall n \in \mathbb{N}, P(n+1)]$ and $[\forall n \in \mathbb{N}, P(n)] \rightarrow [\forall n \in \mathbb{N}, P(n+1)]$. The first parsing is odd: the inner "n" shadows the outer "n", which one wouldn't do intentionally if a formula is meant to be read by a human being (except when teaching logic or as intermediates while applying transformations on logical formulas). So the second one is the most likely.

(c) So (temporarily using (a) to reason about it) $(S1)$ is equivalent to saying: if P is true for all natural numbers then it's true for all natural numbers larger than zero. That sounds true, regardless of which predicate (defined at least on natural numbers) P is.

Proof [proving the implication $[\forall n \in \mathbb{N}, P(n)] \rightarrow [\forall n \in \mathbb{N}, P(n+1)]$]
Assume $\forall n \in \mathbb{N}, P(n)$. [thus if we have a natural number, we can claim P for it]
 [try to prove the universal $\forall n \in \mathbb{N}, P(n+1)$]
 Let $n \in \mathbb{N}$.
  [try to prove $P(n+1)$]
  Then $n+1 \in \mathbb{N}$.
  So by the assumption: $P(n+1)$.
 Since $n \in \mathbb{N}$ was arbitrary: $\forall n \in \mathbb{N}, P(n+1)$.
Thus $\forall n \in \mathbb{N}, P(n) \rightarrow \forall n \in \mathbb{N}, P(n+1)$.

(d) $[\forall n \in \mathbb{N}, P(n+1)] \rightarrow [\forall n \in \mathbb{N}, P(n)]$, i.e. if P is true for all natural numbers larger than zero, then it's true for all natural numbers. This is false for the P shuch that $P(0)$ is false but P is true for all other natural numbers.

2.

(a) $\neg\forall n \in \mathbb{N}, [P(n) \rightarrow P(n+1)]$
$\equiv\ \exists n \in \mathbb{N}, \neg[P(n) \rightarrow P(n+1)]$
$\equiv\ \exists n \in \mathbb{N}, P(n) \wedge \neg P(n+1)$.

(b) $(S0)$ is saying that if P is true for a natural number then it's true for the next one. So (by our belief in induction) if P is ever true, it continues to be true (for the next number, the next number after that, etc). Three kinds of Ps: never true, always true, false for a while then always true (e.g. true for all numbers less than 236, true for all numbers at least 236) [I'll make tables before publishing this for students, but when helping students out with the question: make a blank table and put in T or F somewhere and ask what they can start deducing].

(c) From (a): there's a natural number for which P is true, but false for the next number. E.g., P true for $0, 1, 2$, then false for 3, and arbitrary for the rest.

(d) Only the P that is true for all natural numbers.

(e) In (b) we said: if P is ever true then it continues to be true. P can be false up to some point, but then if it ever becomes true then it continues to be true: $[\exists n \in \mathbb{N}, P(n)] \rightarrow \exists m \in \mathbb{N}, [(\forall k \in \mathbb{N}, k < m \rightarrow \neg P(k)) \wedge (\forall k \in \mathbb{N}, k \geq m \rightarrow P(k))]$. Alternatively, the conjunction in there could be replaced by either implication being turned into an equivalence, e.g. using,$\forall k \in \mathbb{N}, P(k) \leftrightarrow k \geq m$.

3. $(\forall n \in \mathbb{N}, P(n)) \rightarrow (P(0) \wedge \forall n \in \mathbb{N}, P(n) \rightarrow P(n+1))$: if P is true for all natural numbers then it's true for zero and whenever it's true for a natural number it's true for the next natural number.

Proof

Assume $\forall n \in \mathbb{N}, P(n)$. [thus if we have a natural number, we can claim P for it]

[prove $P(0)$ and prove $\forall n \in \mathbb{N}, P(n) \to P(n+1)$]

Since $0 \in \mathbb{N}$: $P(0)$, by the assumption.

Let $n \in \mathbb{N}$, assume $P(n)$.

[try to prove $P(n+1)$]

Then $n+1 \in \mathbb{N}$.

So by assumption in the first line (not even using $P(n)$): $P(n+1)$.

Since $n \in \mathbb{N}$ was arbitrary and we assumed $P(n)$: $\forall n \in \mathbb{N}, P(n) \to P(n+1)$.

So $P(0) \wedge \forall n \in \mathbb{N}, P(n) \to P(n+1)$.

So $(\forall n \in \mathbb{N}, P(n)) \to (P(0) \wedge \forall n \in \mathbb{N}, P(n) \to P(n+1))$.

4.

Proof

Suppose $P(0) \wedge \forall n \in \mathbb{N}, P(n) \to P(n+1)$.

Then $P(0)$.

And $\forall n \in \mathbb{N}, P(n) \to P(n+1)$ (*). [thus if we have a natural number for which $P(n)$, then we can conclude $P(n+1)$]

[prove $P(0) \wedge P(1) \wedge P(2)$]

As noted above: $P(0)$.

Since $0 \in \mathbb{N}$: $P(0) \to P(1)$, by (*). And since $P(0)$ is true, so is $P(1)$.

Since $1 \in \mathbb{N}$: $P(1) \to P(2)$, by (*). And since $P(1)$ is true, so is $P(2)$.

[standard CSC165 backing out to the conclusion, but which we get to drop in CSC236]

5.

(a) $\neg P(0) \vee \exists n \in \mathbb{N}, P(n) \wedge \neg P(n+1)$: P is false for 0 or there's a natural number for which it's true but false for the next natural number.

(b) P is false for 0 but true for all other numbers. Other example: the one from 2(c).

6.

(a) Nothing fancy here, there's a direct recursive translation.

(b) The difficulty here is that simply making $n = 2$ the new base case is insufficient, since $n = 3$ relies on $n = 1$: that's what they're supposed to discover, and use two base cases. In previous years many students simply miss that but ask if there's more to the question: ask them whether they actually tested their function well. To help them discover appropriate base cases in general, have them trace it (in this case) for $n = 2, 3, 4, 5, \ldots$, stating for each number which number(s) it directly relies on: drawing arrows above the sequence, joining each number to the lesser number it relies on is a good visual.