QUESTION 1. [8 MARKS]

Prove that for all natural numbers n greater than 1, the set of the first n positive integers $\{1, \ldots, n\}$ has $3 \cdot 2^{n-2}$ subsets that omit either the element 1, or the element 2, or both the elements 1 and 2.

SAMPLE SOLUTION: Let $P(n)$ be "There are $3 \cdot 2^{n-2}$ subsets of the set of the first n positive integers $\{1, \ldots, n\}$ that omit either the element 1, or the element 2, or both the elements 1 and 2."

PROOF THAT FOR ALL NATURAL NUMBERS GREATER THAN 1, $P(n)$:

BASE CASE: If $n = 2$, the subsets of $\{1, 2\}$ that omit either the element 1, or the element 2, or both the elements 1 and 2 are $\{\}$, $\{1\}$ and $\{2\}$, that is $3 = 3 \cdot 2^{2-2}$ such subsets, so $P(2)$ is true.

INDUCTION STEP: Assume that n is a generic natural number greater than 1, and that $P(n)$ is true. Count the subsets of the set of the first $n + 1$ positive integers that omit either 1, or omit 2, or omit both 1 and 2 by partitioning the subsets into those that include the element $n + 1$, and those that don't (notice that $n + 1 > 2$, so this element is neither 1 nor 2). The relevant subsets that don't include the element $n + 1$ are identical to the relevant subsets of the set $\{1, \ldots, n\}$, and by the IH there are $3 \cdot 2^{n-2}$ of those. The relevant subsets that do include the element $n + 1$ are formed by adjoining element $n + 1$ to each of the relevant subsets that don't include $n + 1$, so there are also $3 \cdot 2^{n-2}$ of these. So, in sum, there are $2 \cdot 3 \cdot 2^{n-2} = 3 \cdot 2^{n-1}$ subsets that omit element 1, or omit element 2, or omit both elements 1 and 2, as claimed by $P(n + 1)$. So, for every natural number n greater than 1, $P(n)$ implies $P(n + 1)$.

I conclude that for every natural number n greater than 1, $P(n)$ is true.

QUESTION 2. [8 MARKS]

Recall the Fibonacci sequence:

$$\forall n \in \mathbb{N} \qquad F(n) = \begin{cases} n, & \text{if } n < 2 \\ F(n-2) + F(n-1), & \text{if } n \geq 2 \end{cases}$$

Prove that for all natural numbers n, $F(n) < 2^n$.

SAMPLE SOLUTION: Define $P(n)$ "$F(n) < 2^n$."

PROOF (COMPLETE INDUCTION) THAT $\forall n \in \mathbb{N}, P(n)$: Assume that n is an arbitrary natural number, and that $P(0) \wedge \cdots \wedge P(n-1)$ are true. If $n = 0$, $F(0) = 0 < 1 = 2^0$, so $P(0)$ is true. If $n = 1$, $F(1) = 1 < 2 = 2^1$, so $P(1)$ is true. Otherwise, $n > 1$ so by the definition of $F(n)$ we have

$$\begin{aligned}
F(n) &= F(n-2) + F(n-1) && \text{(definition of } F(n) \text{ for } n > 1) \\
&< 2^{n-2} + 2^{n-1} && \text{(By } P(n-2) \text{ and } P(n-1) \text{, since } 0 \leq n-2, n-1 \leq n \text{ when } n > 1) \\
&< 2 \times 2^{n-1} && \text{(since } 2^{n-2} < 2^{n-1}) \\
&= 2^n
\end{aligned}$$

So $F(n) < 2^n$, and this is exactly what $P(n)$ says, so $\forall n \in \mathbb{N}$, $P(0) \wedge \cdots P(n-1) \Rightarrow P(n)$.

I conclude that for every natural number n, $P(n)$.

QUESTION 3. [8 MARKS]

Find some natural number k such that for all natural numbers n greater than k, $3^n > 3n^3$. Prove your claim. You may use the binomial expansion, $(n+1)^3 = n^3 + 3n^2 + 3n + 1$.

SAMPLE SOLUTION: Define $P(n)$ as "$3^n > 3n^3$".

PROOF (SIMPLE INDUCTION) THAT FOR ALL NATURAL NUMBERS GREATER THAN 5, $P(n)$:

BASE CASE: If $n = 6$ then $3^6 = 729 > 648 = 3 \times 6^3$, so $P(6)$ is true.

INDUCTION STEP: Assume that n is a generic integer greater than 6, and that $P(n)$ is true. Then

$$\begin{aligned} 3^{n+1} &= 3 \times 3^n > 3 \times 3n^3 \quad \text{(by } P(n)\text{)} \\ &= 3n^3 + 3n^3 + n^3 + 2n^3 \quad \text{(expanding...)} \\ &\geq 3n^3 + 9n^2 + 9n + 3 \quad \text{(Since } 3n^3 \geq 9n^2,\ n^3 \geq 9n, \text{ and } 2n^3 \geq 3 \text{ when } n > 5 \geq 3\text{)} \\ &= 3(n+1)^3 \end{aligned}$$

So $3^{n+1} > 3(n+1)^3$, which is what $P(n+1)$ says, so for every natural number n greater than 5, $P(n) \Rightarrow P(n+1)$.

In conclude that for all natural numbers n greater than 5, $P(n)$.

Total Marks = 24