

PLEASE HAND IN

UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE

TERM TEST #2

CSC 236H1

DURATION — 50 MINUTES

PLEASE HAND IN

LAST NAME: _____

FIRST NAME: _____

Do NOT turn this page until you have received the signal to start.
(In the meantime, please fill out the identification section above,
and read the instructions below.)

This test consists of 2 questions on 3 pages (including this one). *When you receive the signal to start, please make sure that your copy of the test is complete.*

Please answer questions in the space provided. You will earn 20% for any question you leave blank or write "I cannot answer this question," on. You will earn substantial part marks for writing down the outline of a solution and indicating which steps are missing.

Good Luck!

QUESTION 1. [8 MARKS]

The following recurrence is derived from a first attempt divide-and-conquer algorithm for fast multiplication of a pair of n -bit numbers:

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(\lceil n/2 \rceil) + 2T(\lfloor n/2 \rfloor) + n & \text{if } n > 1 \end{cases}$$

Use repeated substitution (aka unwinding) to guess a closed-form expression for $T(n)$ in the special case where n is a power of 2 (i.e. $\exists k \in \mathbb{N}, n = 2^k$). Then use an appropriate flavour of induction to prove your guess is correct.

SOLUTION: It may be clearest to express n as an integer power of 2, that is $n = 2^k$. Then I have:

$$\begin{aligned} T(2^k) &= 2T(\lceil 2^{k-1} \rceil) + 2T(\lfloor 2^{k-1} \rfloor) + 2^k = 4T(2^{k-1}) + 2^k && \# \text{floor, ceiling of an integer are the same} \\ &= 4(4T(2^{k-2}) + 2^{k-1}) + 2^k = 4^2T(2^{k-2}) + 2 \times 2^k + 2^k \\ &= 4^2(4T(2^{k-3}) + 2^{k-2}) + 3 \times 2^k = 4^3T(2^{k-3}) + 4 \times 2^k + 2 \times 2^k + 2^k \\ &\vdots \\ &= 4^i T(2^{k-i}) + \sum_{j=0}^{i-1} 4^j 2^{k-j} = 4^i T(2^{k-i}) + 2^k \sum_{j=0}^{i-1} 2^j = 4^i T(2^{k-i}) + 2^k (2^i - 1) \\ &\vdots \\ &= 4^k T(2^{k-k}) + 2^k (2^k - 1) = 4^{\lg n} \times 1 + n(n-1) \\ &= 2^{2 \lg n} + n^2 - n = 2n^2 - n. \end{aligned}$$

PROOF SOLUTION: Define $P(n)$: "If n is a power of 2, then $T(n) = 2n^2 - n$." I prove $\forall n \in \mathbb{N}, P(n)$ by complete induction.

INDUCTION STEP: Assume $n \in \mathbb{N}$ and that $P(i)$ is true for all natural numbers $i < n$. Consider two cases:

CASE $n < 2$ (BASE CASE): I only need verify $P(0)$, which holds vacuously since 0 is not a power of 2, and $P(1)$ which claims that

$$1 = T(1) = 2 \times 1^2 - 1 = 1$$

Which is certainly true. So the claim holds in this case.

CASE $n \geq 2$: If n is not a power of 2, then $P(n)$ is vacuously true. Otherwise $n \geq 2$, so $n = 2^k$, for some $k \geq 1$, and $n/2 = 2^{k-1} \in \mathbb{N}$, so by assumption I know $P(n/2)$. This means

$$\begin{aligned} T(n) &= 2T(n/2) + 2T(n/2) + n = 4T(n/2) + n \\ &= 4 \left[2(n/2)^2 - n/2 \right] + n && \# \text{ by Induction Hypothesis on } n/2 \\ &= 8(n^2/4) - 2n + n = 2n^2 - n \end{aligned}$$

Thus the claim holds in both cases, so if $n \in \mathbb{N}$ and $P(i)$ is true for all natural numbers $i < n$, then $P(n)$ follows.

I conclude, by the principle of complete induction, $\forall n \in \mathbb{N}, P(n)$.

QUESTION 2. [8 MARKS]

Assume that $G(m) = 2m^2 - m$ whenever m is a power of 2 (i.e. $\exists k \in \mathbb{N}, m = 2^k$), and that G is non-decreasing on positive integers. Prove:

$$\exists c_1, c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 n^2 \leq G(n) \leq c_2 n^2$$

SOLUTION: Let $c_1 = ?$, $c_2 = ??$, and $B = ???$. I will fill in values for c_1 , c_2 , and B in the course of the proof, and it will be clear that $c_1, c_2 \in \mathbb{R}^+$ and $B \in \mathbb{N}$.

Assume $n \in \mathbb{N}^+$, and define $\hat{n} = 2^{\lceil \lg n \rceil}$, the next power of 2 greater than, or equal to, n . Notice the following:

$$\hat{n}/2 = 2^{\lceil \lg n \rceil - 1} < n \leq \hat{n} = 2^{\lceil \lg n \rceil}$$

For both \hat{n} and $\hat{n}/2$ to be powers of 2, I must have $n \geq 2$, so $B \geq 2$. Then,

$$\begin{aligned} G(n) &\leq G(\hat{n}) && \# \text{ Since } G \text{ is non-decreasing and } n \leq \hat{n} \\ &= 2\hat{n}^2 - \hat{n} \leq 2\hat{n}^2 && \# \text{ since } \hat{n} \text{ is a power of 2} \\ &\leq 2(2n)^2 = 8n^2 && \# \text{ since } 2n > \hat{n} \geq 0 \\ &= c_2 n^2 && \# \text{ if } c_2 = 8 \end{aligned}$$

So, if I choose $c_2 = 8$ and $B \geq 2$, then

$$\forall n \in \mathbb{N}, n \geq B \Rightarrow G(n) \leq c_2 n^2$$

Also,

$$\begin{aligned} G(n) &\geq G(\hat{n}/2) && \# \text{ since } G \text{ is non-decreasing} \\ &= 2(\hat{n}/2)^2 - \hat{n}/2 = \frac{\hat{n}^2 - \hat{n}}{2} && \# \text{ since } \hat{n}/2 \text{ is a power of 2} \\ &\geq \frac{\hat{n}^2}{4} \geq \frac{n^2}{4} && \# \text{ since } \hat{n} \geq n \geq B \geq 2 \\ &= c_1 n^2 && \# \text{ if } c_1 = 1/4 \end{aligned}$$

So, if I choose $c_1 = 1/4$ and $B \geq 2$, then

$$\forall n \in \mathbb{N}, n \geq B \Rightarrow G(n) \geq c_1 n^2$$

So, I have shown that $\exists c_1 = 1/4, c_2 = 8 \in \mathbb{R}^+$, and $B = 2 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 n^2 \leq G(n) \leq c_2 n^2$.

1: _____ / 8

2: _____ / 8

TOTAL: _____ / 16