UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE

TERM TEST #1

CSC 236H1
DURATION — 50 MINUTES

LAST NAME: _____

FIRST NAME: _____

Do NOT *turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above,
and read the instructions below.)

This test consists of 3 questions on 7 pages (including this one). *When
you receive the signal to start, please make sure that your copy of the
test is complete.*

Please answer questions in the space provided. You will earn 20% for
any question you leave blank or write "I cannot answer this question,"
on. You will earn substantial part marks for writing down the outline of
a solution and indicating which steps are missing.

Good Luck!

QUESTION 1.   [5 MARKS]

Use Mathematical Induction to prove that for every positive (i.e. greater than 0) natural number n, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

PROOF: For convenience I define the predicate $P(n)$: "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$." I now prove $\forall n \in \mathbb{N}^+, P(n)$ by mathematical induction.

[Base case, n=1:] In this case $\sum_{i=0}^{1} 2^i = 2^0 + 2^1 = 3 = 2^2 - 1$, which verifies $P(1)$.

INDUCTION STEP: Assume n is an arbitrary positive integer and assume $P(n)$. Then

$$
\begin{aligned}
\sum_{i=0}^{n+1} 2^i &= \left(\sum_{i=0}^{n} 2^i \right) + 2^{n+1} &&\text{\# rewrite sum} \\
&= 2^{n+1} - 1 + 2^{n+1} &&\text{\# by IH} \\
&= 2^{n+2} - 1 = 2^{(n+1)+1} - 1
\end{aligned}
$$

Since I assumed $n \in \mathbb{N}^+$ and $P(n)$ and derived $P(n+1)$, I have shown $\forall n \in \mathbb{N}^+, P(n) \Rightarrow P(n+1)$.

I conclude $\forall n \in \mathbb{N}^+, P(n)$, by mathematical induction.

QUESTION 2.   [5 MARKS]

Recall the definition of a full binary tree: a binary tree where each interior node has exactly two children (leaves, by definition, have exactly zero children). Use Complete Induction on the number of nodes, n, to prove that any non-empty full binary tree has exactly one more node than it has edges.

PROOF: For convenience I define the predicate $P(n)$ "Any full binary tree with n nodes has $n - 1$ edges." I now prove $\forall n \in \mathbb{N}, P(n)$ by complete induction.

INDUCTION STEP: Assume $n \in \mathbb{N}$ and assume $P(i)$ for all natural numbers i, $0 \leq i < n$.

CASE $n \leq 1$: The only non-empty full binary tree with no more than 1 node consists of just the root node and no edges. In this case there are $n = 1$ nodes and $n - 1 = 0$ edges, which verifies $P(n)$.

CASE $n > 1$: In this case, the root has at least one child and (being a full binary tree) it has both a left and right child, and each of these are roots of full binary sub-trees. Call these sub-trees T_L and T_R, with n_L and n_R nodes, respectively. Both T_L and T_R have fewer nodes than the original tree (they lack, at least, the root) so we know $0 \leq n_L, n_R < n$, so the sub-trees contribute $n_L - 1$ and $n_R - 1$ edges respectively, by the IH. The total number of nodes in the original tree are comprised of those in each sub-tree plus the root, whereas the total number of edges are comprised of those in each sub-tree plus two connecting the sub-trees to the root. This means that the number of edges in the original tree is

$$n_L - 1 + n_R - 1 + 2 = n_L + n_R = n - 1$$

Since I assumed $n \in \mathbb{N}$ and $P(i)$ whenever $0 \leq i < n$ and then derived $P(n)$, I have shown that $\forall n \in \mathbb{N}, P(0) \wedge \cdots \wedge P(n - 1) \Rightarrow P(n)$.

I conclude $\forall n \in \mathbb{N}, P(n)$ by complete induction.

QUESTION 3. [5 MARKS]

Recall the definition of the Fibonacci sequence:

$$F(n) = \begin{cases} n & \text{if } n < 2 \\ F(n-2) + F(n-1) & \text{if } n \geq 2 \end{cases}$$

Use Mathematical Induction to prove that for every natural number n, $\sum_{i=0}^{n} F(2i+1) = F(2n+2)$.

PROOF: For convenience I define the predicate $P(n)$: "$\sum_{i=0}^{n} F(2i+1) = F(2n+2)$." I prove $\forall n \in \mathbb{N}, P(n)$ by mathematical induction.

BASE CASE: $P(0)$ states that $\sum_{i=0}^{0} F(2 \cdot 0 + 1) = F(1) = F(2 \cdot 0 + 2) = F(2) = 1$, which is certainly true.

INDUCTION STEP: Assume $n \in \mathbb{N}$ and assume $P(n)$. Then

$$\begin{aligned} \sum_{i=0}^{n+1} F(2i+1) &= \left(\sum_{i=0}^{n} F(2i+1) \right) + F(2(n+1)+1) \qquad \text{\# rewrite sum} \\ &= F(2n+2) + F(2n+3) \qquad \text{\# by IH} \\ &= F(2n+4) = F(2(n+1)+2) \qquad \text{\# definition of } F(2n+4), 2n+4 > 1 \end{aligned}$$

Since I assumed n was a generic natural number and $P(n)$ and then derived $P(n+1)$, I have shown $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$.

Given the base case and the induction step, by the principle of mathematical induction I conclude $\forall n \in \mathbb{N}, P(n)$.

Please note that in spite of the definition of $F(n)$ having 2 base cases, this proof requires only one.

\# 1: _____/ 5

\# 2: _____/ 5

\# 3: _____/ 5

TOTAL: _____/15