PLEASE HAND IN

UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE

TERM TEST #1

CSC 236H1
DURATION — 50 MINUTES

PLEASE HAND IN

LAST NAME: ______________________________

FIRST NAME: ______________________________

Do NOT *turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below.)

This test consists of 3 questions on 7 pages (including this one). *When you receive the signal to start, please make sure that your copy of the test is complete.*

Please answer questions in the space provided. You will earn 20% for any question you leave blank or write "I cannot answer this question," on. You will earn substantial part marks for writing down the outline of a solution and indicating which steps are missing.

Good Luck!

QUESTION 1. [5 MARKS]

Use Mathematical Induction to prove that for every natural number n, $13^n - 1$ is a multiple of 12.

PROOF: For convenience I define $P(n)$: "$\exists z \in \mathbb{Z}, 13^n - 1 = 12z$" I use Mathematical Induction to prove $\forall n \in \mathbb{N}, P(n)$.

BASE CASE, N=0: If $n = 0$ then $13^n - 1 = 13^0 - 1 = 0$ and $0 = 12 \times 0, 0 \in \mathbb{Z}$, so $P(0)$ is verified.

INDUCTION STEP: Assume $n \in \mathbb{N}$ and $P(n)$. Then

$$\begin{aligned} 13^{n+1} - 1 &= 13(13^n - 1) + 12 \qquad \# \text{ rewrite} \\ &= 13(12z) + 12, \text{ some } z \in \mathbb{Z} \qquad \# \text{ by } P(n) \\ &= 12(13z + 1) = 12z', z' \in \mathbb{Z} \qquad \# \ 13, z, 1 \in \mathbb{Z} \\ &\qquad \# \text{ and } \mathbb{Z} \text{ is closed under } +, \times \end{aligned}$$

Since I assume n was an arbitrary natural number and $P(n)$, and from that derived $P(n+1)$, then $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$.

I conclude, by mathematical induction, $\forall n \in \mathbb{N}, P(n)$.

QUESTION 2. [5 MARKS]

Recall the definition of a binary tree (a tree where each node has, at most, two children). Define the height of a binary tree as the number of nodes in the longest path from root to a leaf — so a tree consisting of a single node has height 1. Use Complete Induction to prove that for every natural number n, any non-empty binary tree of height n has, at most, 2^{n-1} leaves (leaves are nodes with zero children).

SOLUTION: For convenience, I define the predicate $P(n)$: "Any non-empty binary tree of height n has, at most 2^{n-1} leaves." I use complete induction to prove $\forall n \in \mathbb{N}, P(n)$.

INDUCTION STEP: Assume $n \in \mathbb{N}$, and that $P(i)$ is true for all $0 \leq i < n$.

CASE $n \leq 1$: The only non-empty binary tree with height no greater than 1 consists of a childless root, and has height 1. Since the root is also the sole leaf, this tree has no more than $2^{1-1} = 1$ leaf, so $P(1)$ is verified (and $P(0)$ is vacuously verified, since there are no non-empty binary trees of height 0).

CASE $n > 1$: A tree with height 2 or more has either 1 or two non-empty sub-trees, and these contribute all t he leaves. I consider the cases separately.

SUB-CASE A, EXACTLY ONE NON-EMPTY SUB-TREE: In this case, the single sub-tree has height $n-1$ (the longest path, minus the root node), and by the IH it has no more than 2^{n-1-1} leaves, which (in turn) is no more than 2^{n-1} leaves.

SUB-CASE B, EXACTLY TWO NON-EMPTY SUB-TREES: Call the height of the left sub-tree n_L, and that of the right sub-tree n_R. Since they are non-empty, $1 \leq n_L, n_R$, and since the longest path in each sub-tree is shorter (by one) than the corresponding path that includes the root of the original tree, they have $n_L, n_R \leq n-1$. This means that the the IH applies, and the total number of leaves contributed by the two sub-trees is

$$2^{n_L-1} + 2^{n_R-1} \leq 2^{n-1-1} + 2^{n-1-1} = 2^{n-1}$$

So $\forall n \in \mathbb{N}$, if $P(i)$ for $0 \leq i < n$, then $P(n)$.

Then, by complete induction, I conclude $\forall n \in \mathbb{N}, P(n)$.

(Another approach would be to strengthen the claim so that it applied to all binary trees (not just non-empty ones), and the claim about non-empty binary trees would follow as a consequence).

QUESTION 3. [5 MARKS]

Define $\mathcal{E}$ as the SMALLEST set such that

1. Symbols x, y, and z are elements of $\mathcal{E}$.
2. If expressions e_1 and e_2 are elements of $\mathcal{E}$, then so is $(e_1 + e_2)$.

For $e \in \mathcal{E}$ define $\mathrm{vr}(e)$ as the number of occurrences of symbols from the set $\{x, y, z\}$ in e, and $\mathrm{p}(e)$ as the number of occurrences of parentheses from the set $\{(,)\}$ in e. For example, $\mathrm{vr}(((x + y) + z)) = 3$ and $\mathrm{p}(((x + y) + z)) = 4$. Use Structural Induction to prove that for every $e \in \mathcal{E}$, $2\mathrm{vr}(e) = \mathrm{p}(e) + 2$.

SOLUTION: for convenience, I define the predicate $P(e)$: "$2\mathrm{vr}(e) = \mathrm{p}(e) + 2$" I use structural induction to prove $\forall e \in \mathcal{E}, P(e)$.

BASIS, $e \in \{x, y, z\}$: Each of these expressions has exactly one variable and 0 parentheses, and $2 \times 1 = 0+2$, so $P(e)$ is verified for e in the basis.

INDUCTION STEP: Assume $e_1, e_2 \in \mathcal{E}$ and $P(e_1)$, $P(e_2)$. Then

$$\begin{aligned}
2\mathrm{vr}((e_1 + e_2)) &= 2\,(\mathrm{vr}(e_1) + \mathrm{vr}(e_2)) \qquad \text{\# no variables added} \\
&= 2\mathrm{vr}(e_1) + 2\mathrm{vr}(e_2) \\
&= \mathrm{pr}(e_1) + 2 + \mathrm{pr}(e_2) + 2 = [\mathrm{pr}(e_1) + \mathrm{pr}(e_2) + 2] + 2 \qquad \text{\# by IH} \\
&= \mathrm{pr}((e_1 + e_2)) + 2 \qquad \text{\# 2 parentheses needed to wrap } (e_1 + e_2)
\end{aligned}$$

So, $\forall e_1, e_2 \in \mathcal{E},\ P(e_1) \wedge P(e_2) \Rightarrow P((e_1 + e_2))$.

By the principle of structural induction, I conclude $\forall e \in \mathcal{E}, P(e)$.

1: _______/ 5

2: _______/ 5

3: _______/ 5

TOTAL: _______/15