CSC236 tutorial exercises #4

(Best before 11 am, Monday October 22nd)

Danny Heap

Here are your tutorial sections:

Surname	Section	Room	TA
A–F	Day 1 (11:00 am)	LM162	Yuval
G–Li	Day 2 (11:00 am)	BA2139	Lila
Lo–Si	Day 3 (11:00 am)	BA2145	Oles
So–Z	Day 4 (11:00 am)	BA2155	Lalla
A–H	Evening 1 (8:00 pm)	BA1190	Colin
I–M	Evening 2 (8:00 pm)	BA2135	Norman
N–Z	Evening 3 (8:00 pm)	BA2139	Feyyaz

These exercises are meant to give you practice with some of the concepts used to prove the Master Theorem.

1. Consider the recurrence:

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + n + 1 & \text{if } n > 1 \end{cases}$$

This recurrence is superficially different from the one derived in the Course notes. Use the above recurrence and the approach of Lemma 3.7 in the Course Notes to show that T is non-decreasing.

Claim: Define $P(n)$: for every positive integer m, $m < n \Rightarrow T(m) \leq T(n)$. I use complete induction to prove that $\forall n \in \mathbb{N}^+, P(n)$.

Induction step: Assume that n is an arbitrary positive integer, and that $P(k)$ is true for $1 \leq k < n$.

Case $1 \leq n < 3$: $P(1)$ is vacuously true, since there are no positive integers less than 1. To establish $P(2)$ I calculate $T(1) = 1$ and $T(2) = 2T(1) + 2 + 1 = 5$, and note that $1 \leq 5$, so $P(1)$ and $P(2)$ are each verified.

Case $n > 2$: Then $1 \leq \lfloor n/2 \rfloor \leq \lceil n/2 \rceil < n$, by Lemma 3.8. Also $1 \leq n - 1 < n$, so $P(n-1)$ is true, by assumption, and the only thing left is to show $T(n-1) \leq T(n)$, and

$$\begin{aligned} T(n-1) &= T(\lceil (n-1)/2 \rceil) + T(\lfloor (n-1)/2 \rfloor) + (n-1) + 1 \qquad \# \text{ apply definition} \\ &\leq T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + n + 1 \qquad \# \text{ by } P(\lfloor n/2 \rfloor), P(\lceil n/2 \rceil) \\ &\qquad \# \text{ also } n - 1 \leq n \text{ and } 1 \leq \lfloor (n-1)/2 \rfloor \leq \lfloor n/2 \rfloor \\ &= T(n) \end{aligned}$$

Conclude $\forall n \in \mathbb{N}^+, P(n)$ by complete induction.

2. Use repeated substitution (unwinding) to find a closed form for the recurrence S when n is a power of 3:

$$S(n) = \begin{cases} 1 & \text{if } n < 3 \\ a_1 S(\lceil n/3 \rceil) + a_2 S(\lfloor n/3 \rfloor) + n^2 & \text{if } n > 2 \end{cases}$$

...where integers $a_1, a_2 \geq 0$ and $a_1 + a_2 = 3$.

Solution: If n is an integer power of 3 greater than 3^0, then $\lfloor n/3 \rfloor$ is the same as $\lceil n/3 \rceil$, and the recurrence can be simplified to:

$$S(n) = 3S(n/3) + n^2$$

Unwind this a few steps to see a pattern:

$$\begin{aligned}
S(n) &= 3S(n/3) + n^2 \\
&= 3(3S(n/9) + (n/3)^2) + n^2 \\
&= 3^2 S(n/9) + n^2/3 + n^2 \\
&= 3^2(3S(n/27) + (n/9)^2) + n^2/3 + n^2 \\
&= 3^3 S(n/27) + n^2/9 + n^2/3 + n^2 \\
&\vdots \\
&= 3^k S(n/n) + n^2 \sum_{i=0}^{k-1} 1/3^i \qquad \#k = \log_3 n \\
&= n + n^2 \frac{1-(1/3)^k}{1-1/3} \qquad \#\text{formula for geometric series} \\
&= n + n^2 \frac{(3^k-1)/3^k}{2/3} = n + \frac{3}{2}\frac{n^2(n-1)}{n} \qquad \#n = 3^k \\
&= n + \frac{3}{2}n(n-1)
\end{aligned}$$

Claim: $\forall k \in \mathbb{N}, S(3^k) = 3^k + \frac{3}{2}3^k(3^k - 1)$. For convenience, define $P(k) : S(3^k) = 3^k + \frac{3}{2}3^k(3^k - 1)$.

Proof (by mathematical induction):

Base case $k = 0$: By definition $S(3^0) = 1$, and that's also equal to $3^0 + \frac{3}{2}3^0(3^0 - 1)$, as claimed. So the $P(0)$ is true.

Induction step: Assume that k is an arbitrary natural number and assume that $P(k)$ is true. Then

$$\begin{aligned}
S(3^{k+1}) &= 3S(3^k) + (3^{k+1})^2 \qquad \#\text{ apply definition for } 3^{k+1} > 2 \\
&= 3\left(3^k + \frac{3}{2}3^k(3^k - 1)\right) + (3^{k+1})^2 \qquad \#\text{ apply IH to } S(3^k) \\
&= 3^{k+1} + \frac{3}{2}3^{k+1}(3^k - 1) + (3^{k+1})^2 \\
&= 3^{k+1} + 3^{k+1}\frac{3^{k+1} - 3 + 2 \times 3^{k+1}}{2} \qquad \#\text{ factor out } 3^{k+1} \\
&= 3^{k+1} + \frac{3}{2}3^{k+1}(3^{k+1} - 1)
\end{aligned}$$

...So $S(3^{k+1}) = 3^{k+1} + \frac{3}{2}3^{k+1}(3^{k+1} - 1)$.

So, $\forall k \in \mathbb{N}$, $P(k) \Rightarrow P(k+1)$.

Conclude, $\forall k \in \mathbb{N}, P(k)$, by mathematical induction.